Solutions to Review Problems for Exam 1

1. Sketch the curve C parametrized by

$$\begin{cases} x = \sin^2(t); \\ y = \cos^2(t), \end{cases} \qquad \text{for } 0 \leqslant t \leqslant \frac{\pi}{2}. \tag{1}$$

***Solution**:* Since $\cos^2 t + \sin^t = 1$, for all $t \in \mathbb{R}$, we obtain from the parametric equations in (1) that

$$x + y = 1. \tag{2}$$

Thus, the curve C lies on the straight line given by the equation in (2). To find out which portion of the line in (2) the parametric equations in (1) represent, note that, as t goes from 0 to $\frac{\pi}{2}$, the x–coordinates of points in C range from 0 to 1. Similarly, the y–coordinates of points on C range from 1 to 0. Consequently,

$$C = \{(x, y) \in \mathbb{R}^2 \mid x + y = 1 \text{ and } 0 \leqslant x \leqslant 1\}.$$

C is sketched in Figure 1. □

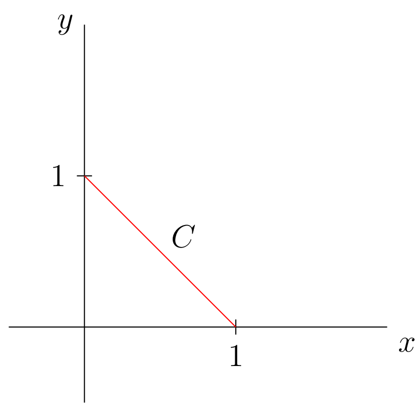

Figure 1: Sketch of C

2. A curve C is parametrized by the differentiable path given by

$$\sigma(t) = (3t^2, 2 + 5t), \qquad \text{for } t \in \mathbb{R}.$$

Sketch the curve C in the xy–plane. Describe the curve.

Solution: The parametric equations of C are

$$\begin{cases} x = 3t^2; \\ y = 2 + 5t. \end{cases} \tag{3}$$

Solving for t in the second equation in (3) yields

$$t = \frac{y-2}{5},$$

and substituting into the first equation

$$x = 3\left(\frac{y-2}{5}\right)^2,$$

or

$$x = \frac{3}{25}(y-2)^2. \tag{4}$$

The graph of the equation in (4) is a parabola with vertex at $(0, 2)$, which opens up to the the right; see the sketch in Figure 2. □

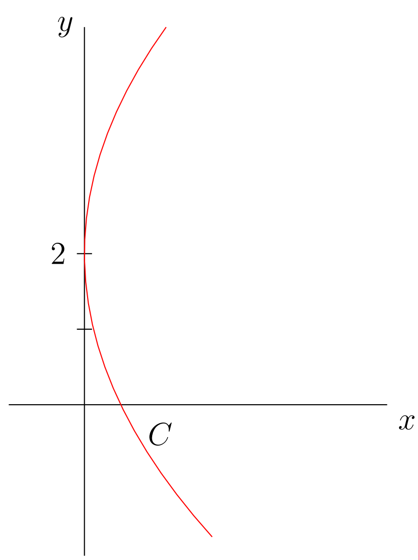

Figure 2: Sketch of parabola C

3. Sketch the curve C parametrized by

$$\begin{cases} x = 2 + 3\cos t; \\ y = 1 + \sin t, \end{cases} \quad \text{for } 0 \leqslant t \leqslant 2\pi. \tag{5}$$

Describe the curve.

Solution: From the parametric equations in (5) we obtain

$$\frac{x-2}{3} = \cos t \quad \text{and} \quad y - 1 = \sin t,$$

from which we get that

$$\left(\frac{x-2}{3}\right)^2 + (y-1)^2 = 1,$$

or

$$\frac{(x-2)^2}{9} + (y-1)^2 = 1. \tag{6}$$

The graph of the equation in (6) is an ellipse centered at the point $(2, 1)$ with major parallel to the x–axis and of length 6, and its minor axis parallel to the y–axis and of length 2. This ellipse is shown in Figure 3. □

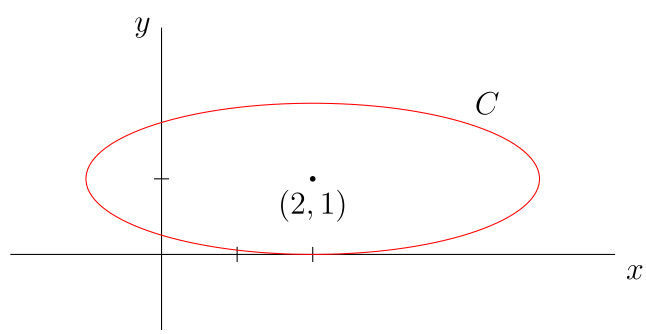

Figure 3: Sketch of Ellipse

4. Give a parametrization for the portion of the circle of radius 2 centered at $(1, 1)$ from the point $P(1, 3)$ to the point $Q(3, 1)$.

Solution: The equation of the circle of radius 2 and center at $(1, 1)$ in Cartesian coordinates is

$$(x-1)^2 + (y-1)^2 = 4, \tag{7}$$

from which we get that

$$\frac{(x-1)^2}{4} + \frac{(y-1)^2}{4} = 1,$$

or

$$\left(\frac{x-1}{2}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1. \tag{8}$$

Setting

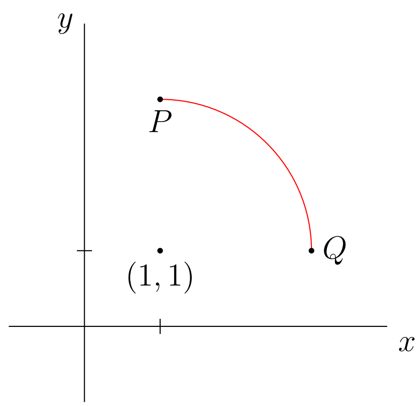

Figure 4: Sketch of C

$$\frac{x-1}{2} = \sin t \quad \text{and} \quad \frac{y-1}{2} = \cos t,$$

we see that the equation in (8) is satisfied. We therefore get the parametric equations

$$\begin{cases} x = 1 + 2\sin t; \\ y = 1 + 2\cos t. \end{cases} \tag{9}$$

To get the portion of the circle in (7) that goes from the point P go the point Q pictured in Figure 4, we restrict t in (9) to go from 0 to $\frac{\pi}{2}$. □

5. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ denote distinct points in the plane. Give a parametrization of the directed line segment $\overrightarrow{PQ}$.

***Solution**:* Figure 5 shows the situation in which x_1, x_2, y_1 and y_2 are positive, and $x_1 < x_2$ and $y_1 < y_2$.

Define the vector

$$v = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}.$$

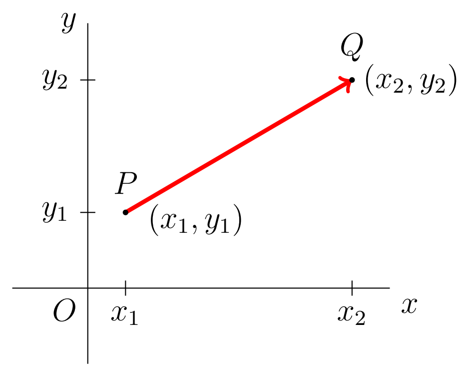

Figure 5: Sketch of directed line segment from P to Q

Then, the vector–parametric equation of the directed line segment $\overrightarrow{PQ}$ is given by

$$\sigma(t) = \overrightarrow{OP} + tv, \quad \text{for } 0 \leqslant t \leqslant 1. \tag{10}$$

The vector–parametric equation in (10) can be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}, \quad \text{for } 0 \leqslant t \leqslant 1,$$

or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} (x_2 - x_1)t \\ (y_2 - y_1)t \end{pmatrix}, \quad \text{for } 0 \leqslant t \leqslant 1,$$

or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 + (x_2 - x_1)t \\ y_1 + (y_2 - y_1)t \end{pmatrix}, \quad \text{for } 0 \leqslant t \leqslant 1. \tag{11}$$

The vector equation in (11) is equivalent to the parametric equations

$$\begin{cases} x = x_1 + (x_2 - x_1)t; \\ y = y_1 + (y_2 - y_1)t, \end{cases} \quad \text{for } 0 \leqslant t \leqslant 1.$$

□

6. Given a curve C parametrized by a differentiable path $\sigma \colon J \to \mathbb{R}^2$, where J is an open interval, the tangent line to the curve at the point $\sigma(t_o)$, where $a < t_o < b$, is the straight line through $\sigma(t_o)$ in the direction of $\sigma'(t_o)$. The vector–parametric equation of this line is given by

$$\ell(t) = \sigma(t_o) + (t - t_o)\sigma'(t_o), \quad \text{for } t \in \mathbb{R}.$$

For the given parametrizations, give the vector–parametric equation of the tangent line to the path at the indicated point.

(a) $\sigma(t) = t\widehat{i} + t^2\widehat{j}$, for $t \in \mathbb{R}$, at the point $(1, 1)$.

***Solution**:* The point $(1, 1)$ corresponds to $t_o = 1$. Thus, the vector–parametric equation of the tangent line to the curve parametrized by σ at the point $(1, 1)$ is

$$\ell(t) = \sigma(1) + (t - 1)\sigma'(1), \quad \text{for } t \in \mathbb{R},$$

where

$$\sigma'(t) = \hat{\imath} + 2t\hat{\jmath}, \quad \text{for } t \in \mathbb{R};$$

so that,

$$\sigma'(1) = \hat{\imath} + 2\hat{\jmath}.$$

Thus, the vector–parametric equation of the tangent line to the path σ at $\sigma(1) = \hat{\imath} + \hat{\jmath}$ is

$$\ell(t) = \hat{\imath} + \hat{\jmath} + (t - 1)(\hat{\imath} + 2\hat{\jmath}), \quad \text{for } t \in \mathbb{R},$$

or

$$\ell(t) = t\hat{\imath} + (1 + 2(t - 1))\hat{\jmath} \quad \text{for } t \in \mathbb{R},$$

or

$$\ell(t) = t\hat{\imath} + (2t - 1)\hat{\jmath} \quad \text{for } t \in \mathbb{R}.$$

□

(b) $\sigma(t) = \begin{pmatrix} 2t - t^2 \\ t^2 \end{pmatrix}$, for$t \in \mathbb{R}$, at the point $(0, 4)$.

***Solution**:* The point $(0, 4)$ corresponds to $t_o = 2$. Thus, the vector–parametric equation of the tangent line to the path σ at the point $(0, 4)$ is

$$\ell(t) = \sigma(2) + (t - 2)\sigma'(2), \quad \text{for } t \in \mathbb{R},$$

where

$$\sigma'(t) = \begin{pmatrix} 2 - 2t \\ 2t \end{pmatrix}, \quad \text{for } t \in \mathbb{R};$$

so that

$$\sigma'(2) = \begin{pmatrix} -2 \\ 4 \end{pmatrix}.$$

Thus, the vector–parametric equation of the tangent line to σ at the point $\sigma(2)$ is

$$\ell(t) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + (t-2)\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

or

$$\ell(t) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} -2(t-2) \\ 4(t-2) \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

which simplifies to

$$\ell(t) = \begin{pmatrix} 4-2t \\ 4t-4 \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

□

7. Let C denote the unit circle in the xy–plane centered at the origin. Give the coordinates of the points on C at which the tangent line is parallel to the line $y = x$.

***Solution**:* Parametrize C with the path $\sigma\colon [0, 2\pi) \to \mathbb{R}^2$ given by

$$\sigma(t) = \cos t\hat{i} + \sin t\hat{j}, \quad \text{for } 0 \leqslant t < 2\pi.$$

A tangent vector to this path at $\sigma(t)$ is given by

$$\sigma'(t) = -\sin t\hat{i} + \cos t\hat{j}, \quad \text{for } 0 < t < 2\pi. \tag{12}$$

We want to find t so that the vector in (12) is parallel to the line $y = x$, which is parametrized by the parametric equations

$$\begin{cases} x = t; \\ y = t \end{cases}, \quad \text{for } t \in \mathbb{R}.$$

Thus, a direction vector of the line $y = x$ is

$$v = \hat{i} + \hat{j}. \tag{13}$$

For the vector $\sigma'(t)$ in (12) to be parallel to v in (13) there must be a nonzero scalar λ such that

$$\sigma'(t) = \lambda v,$$

or

$$-\sin t\hat{i} + \cos t\hat{j} = \lambda(\hat{i} + \hat{j}),$$

or

$$-\sin t\hat{\imath} + \cos t\hat{\jmath} = \lambda\hat{\imath} + \lambda\hat{\jmath},$$

from which we get

$$-\sin t = \lambda \quad \text{and} \quad \cos t = \lambda. \tag{14}$$

It follows from the equations in (14) and the fact that $\cos^2 t + \sin^2 t = 1$ that

$$\lambda^2 + \lambda^2 = 1,$$

or

$$2\lambda^2 = 1,$$

or

$$\lambda^2 = \frac{1}{2}.$$

We therefore get two possibilities for λ:

$$\lambda_1 = \frac{\sqrt{2}}{2} \quad \text{and} \quad \lambda_1 = -\frac{\sqrt{2}}{2}.$$

For $\lambda_1 = \dfrac{\sqrt{2}}{2}$, we get from (14) that

$$\cos t = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin t = -\frac{\sqrt{2}}{2}.$$

This corresponds to a value of t given by

$$t_1 = \frac{7\pi}{4} \tag{15}$$

On the other hand, if $\lambda_1 = -\dfrac{\sqrt{2}}{2}$, the equations in (14) yield

$$\cos t = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin t = \frac{\sqrt{2}}{2}.$$

This corresponds to a value of t given by

$$t_2 = \frac{3\pi}{4}. \tag{16}$$

Thus, the points on the circle C at which the tangent lines are parallel to the line $y = x$ are $\sigma(t_1)$, where t_1 is given in (15), and $\sigma(t_2)$, where t_2 is given in (16). This yields points on C with coordinates

$$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

and

$$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),$$

respectively. □

8. Given a differentiable path, $\sigma\colon J \to \mathbb{R}^2$, where J is an open interval, the linear approximation of $\sigma(t)$, for t near $t_o \in J$, is the vector–valued function

$$\ell(t) = \sigma(t_o) + (t - t_o)\sigma'(t_o), \quad \text{for } t \in \mathbb{R}.$$

Give the linear approximations to the paths at the indicated points

(a) $\sigma(t) = (t^3, 2 + t^2)$, for $t \in \mathbb{R}$, at the point $(1, 3)$.
***Solution**:* The point $(1, 3)$ corresponds to $t_o = 1$.
The linear approximation to σ for t near 1 is

$$\ell(t) = \sigma(1) + (t - 1)\sigma'(1), \quad \text{for } t \in \mathbb{R},$$

where

$$\sigma'(t) = 3t^2\hat{\imath} + 2t\hat{\jmath}, \quad \text{for } t \in \mathbb{R}.$$

Thus, in particular,

$$\sigma'(t) = 3\hat{\imath} + 2\hat{\jmath}.$$

We then have that the linear approximation to $\sigma(t)$, for t near 1 is

$$\begin{aligned} \ell(t) &= \hat{\imath} + 3\hat{\jmath} + (t - 1)(3\hat{\imath} + 2\hat{\jmath}) \\ &= [1 + 3(t - 1)]\hat{\imath} + [3 + 2(t - 1)]\hat{\jmath} \end{aligned}$$

for t near 1, which simplifies to

$$\ell(t) = (3t - 2)\hat{\imath} + (2t + 1)\hat{\jmath}, \quad \text{for } t \text{ near } 1.$$

□

(b) $\sigma(t) = (t, t - t^3)$, for $t \in \mathbb{R}$, at the point $(1, 0)$.
***Solution**:* The point $(1, 0)$ corresponds to $t_o = 1$.
The linear approximation to σ for t near 1 is

$$\ell(t) = \sigma(1) + (t-1)\sigma'(1), \quad \text{for } t \in \mathbb{R},$$

where

$$\sigma'(t) = \hat{\imath} + (1 - 3t^2)\hat{\jmath}, \quad \text{for } t \in \mathbb{R}.$$

Thus, in particular,

$$\sigma'(t) = \hat{\imath} - 2\hat{\jmath}.$$

We then have that the linear approximation to $\sigma(t)$, for t near 1 is

$$\begin{aligned} \ell(t) &= \hat{\imath} + (t-1)(\hat{\imath} - 2\hat{\jmath}) \\ &= t\hat{\imath} - 2(t-1)\hat{\jmath} \end{aligned}$$

for t near 1, or

$$\ell(t) = t\hat{\imath} + (2 - 2t)\hat{\jmath}, \quad \text{for } t \text{ near } 1.$$

□

9. The line L_1 is given by the parametric equations

$$\begin{cases} x = 1 + 2t; \\ y = 3 - t, \end{cases} \quad \text{for } t \in \mathbb{R}, \tag{17}$$

and the line L_2 is given by the parametric equations

$$\begin{cases} x = 3s; \\ y = 1 + s, \end{cases} \quad \text{for } s \in \mathbb{R}, \tag{18}$$

where t and s are parameters.

(a) Determine whether or not the lines L_1 and L_2 meet. Explain the reasoning leading to your answer.
***Solution**:* Line L_1 has a vector–parametric equation

$$\ell_1(t) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + tv_1, \quad \text{for } t \in \mathbb{R},$$

where

$$v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{19}$$

is a direction vector of L_1.

Similarly, the vector–parametric equation of L_2 is

$$\ell_2(s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + sv_2, \quad \text{for } s \in \mathbb{R},$$

where

$$v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{20}$$

is a direction vector of L_2.

Since v_1 is not a scalar multiple of v_2, the lines L_1 ans L_2 are not parallel. Hence, they must intersect somewhere. □

(b) If the lines L_1 and L_2 do meet, determine the point where they intersect, and give the cosine of the angle the two lines make at the point of intersection.

***Solution**:* To find the point of intersection of L_1 and L_2, set corresponding components in the parametric equations in (17) and (18) equal to each other to get the system equations

$$\begin{cases} 1 + 2t &= 3s; \\ 3 - t &= 1 + s, \end{cases}$$

or

$$\begin{cases} 2t - 3s &= -1; \\ t + s &= 2. \end{cases} \tag{21}$$

The system in (21) can be solved simultaneously to yield $t = 1$ and $s = 1$. Hence, the lines L_1 and L_2 meet at the point $\ell_(1) = \ell_2(1) = (3, 2)$.

The cosine of the angles between the lines at the point they intersect is given by

$$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\|\|v_2\|},$$

where v_1 and v_2 are the direction vectors of L_1 and L_2, respectively, given in (19) and (20), respectively.

Thus,

$$v_1 \cdot v_2 = (2)(3) + (-1)(1) = 5,$$

$$\|v_1\| = \sqrt{2^2 + (-1)^2} = \sqrt{5},$$

and

$$\|v_2\| = \sqrt{3^2 + 1^2} = \sqrt{10}.$$

Consequently,

$$\cos\theta = \frac{5}{\sqrt{5}\sqrt{10}},$$

or

$$\cos\theta = \frac{1}{\sqrt{2}},$$

or

$$\cos\theta = \frac{\sqrt{2}}{2}.$$

□

10. A curve C in the plane is given by the parametric equations

$$\left\{\begin{array}{rcl} x & = & e^t; \\ y & = & e^{-2t}, \end{array}\right. \qquad \text{for } t \in \mathbb{R}. \tag{22}$$

(a) Sketch the curve C in the xy–plane and indicated the direction along the curve given by the parametrization.

Solution*:* We first note that, since the exponential function is always positive, we get from the parametric equations in (22) that $x > 0$ and $y > 0$. Consequently, the curve C lies entirely in the first quadrant.

Squaring on both sides of the first equation in (22) we see that

$$x^2 = e^{2t}, \qquad \text{for } t \in \mathbb{R}.$$

Comparing this equation with the second equation in (22) we also see that

$$x^2 y = 1,$$

from which we get that

$$y = \frac{1}{x^2}, \qquad \text{with } x > 0. \tag{23}$$

A sketch of the graph of the equation in (23 is shown in Figure 6. □

(b) Verify that the point $(1, 1)$ is on the curve C. Explain your reasoning.

Solution*:* Note that the point $(1, 1)$ corresponds to $t = 0$ is the parametric equations in (22). Thus, the point $(1, 1)$ is on the curve C. □

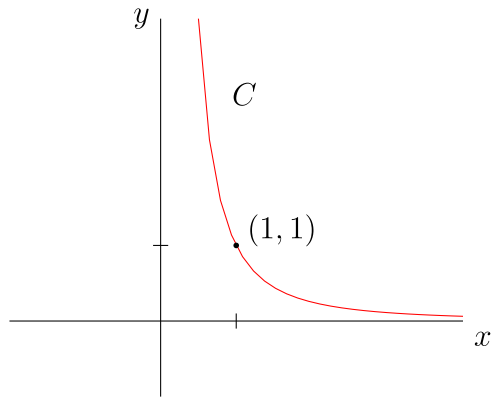

Figure 6: Sketch of C

(c) Give the vector–parametric equation of the tangent line to the curve at the point $(1, 1)$.

***Solution**:* The parametric equations in (22) define a parametrization $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ for C given by

$$\sigma(t) = e^t\ \hat{i} + e^{-2t}\hat{j}, \qquad \text{for } t \in \mathbb{R}. \tag{24}$$

Since the point $(1, 1)$ corresponds to $t_o = 0$, the vector–parametric equation of the tangent line to the path σ defined in (24) is

$$\ell(t) = \sigma(0) + t\sigma'(0), \qquad \text{for } t \in \mathbb{R},$$

where, according to (24),

$$\sigma'(0) = \hat{i} - 2\hat{j}. \tag{25}$$

Then, he vector–parametric equation of the tangent line to the curve C at the point $(1, 1)$ is

$$\ell(t) = \hat{i} + \hat{j} + t(\hat{i} - 2\hat{j}), \qquad \text{for } t \in \mathbb{R},$$

or

$$\ell(t) = (1 + t)\hat{i} + (1 - 2t)\hat{j}, \qquad \text{for } t \in \mathbb{R}.$$

□

(d) Give the vector–parametric equation of the line perpendicular to the tangent line to the curve at the point $(1, 1)$.

***Solution**:* A vector–parametric equation of a line perpendicular to the tangent line to the curve C at the point $(1, 1)$ is

$$p(t) = \hat{i} + \hat{j} + tv, \quad \text{for } t \in \mathbb{R},$$

where v is a vector that is perpendicular to $\sigma'(0)$ given in (25). Thus, we may take

$$v = 2\hat{i} + \hat{j}.$$

Consequently,

$$p(t) = \hat{i} + \hat{j} + t(2\hat{i} + \hat{j}), \quad \text{for } t \in \mathbb{R},$$

or

$$p(t) = (1 + 2t)\hat{i} + (1 + t)\hat{j}, \quad \text{for } t \in \mathbb{R}.$$

□