Review Problems for Exam 3

1. For the linear system of differential equations

$$\begin{cases} \dot{x} &= y; \\ \dot{y} &= -2x - 3y, \end{cases}$$

   (a) compute and sketch line–solutions, if any;

   (b) sketch the nullclines;

   (c) sketch the phase portrait of the system;

   (d) describe the nature of the stability (or unstability) of the origin.

2. Let $f(x, y) = x^2 - y^2$ for all $(x, y) \in \mathbb{R}^2$, and let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field $F(x, y) = \nabla f(x, y)$, for all $(x, y) \in \mathbb{R}^2$.

   (a) Sketch a contour plot for the function f.

   (b) Compute and sketch the flow of the vector field F.

3. Give the formula for an affine function $f\colon \mathbb{R}^2 \to \mathbb{R}$ whose graph contains the points $(1, 4, 7)$, $(4, 7, 0)$ and $(0, 4, 7)$. Sketch the graph of f.

4. Assume that the temperature, $T(x, y)$, at a point (x, y) in the plane is given by

$$T(x, y) = \frac{100}{1 + x^2 + y^2}, \qquad \text{for all } (x, y) \in \mathbb{R}^2.$$

   (a) Sketch the contour plot for T.

   (b) Locate the hottest point in the plane. What is the temperature at that point?

   (c) Give the direction of greatest increase in temperature at the point $(1, 1)$. What is the rate of change of temperature in that direction?

   (d) A bug moves in the plane along a path given by $\sigma(t) = t\,\hat{\imath} + t^2\,\hat{\jmath}$ for $t \in \mathbb{R}$. How fast is the temperature changing when $t = 1$?

5. For the linear system of differential equations

$$\begin{cases} \dot{x} &= x + y - 1; \\ \dot{y} &= -x + y, \end{cases}$$

   (a) sketch the nullclines and find the equilibrium points;

   (b) sketch the phase portrait of the system;

   (c) describe the nature of the stability (or unstability) of the equilibrium points.

6. Sketch the flow of the linear vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x,y) = (6x + 4y)\,\hat{\imath} - (10x + 6y)\,\hat{\jmath} \quad \text{for } (x,y) \in \mathbb{R}^2.$$

   Suggestion: Sketch nullclines and determine the nature of the stability of the origin.

7. Let $f(x,y) = \dfrac{x+y}{1+x^2}$ for all $(x,y) \in \mathbb{R}^2$. Compute the rate of change of f at $(1,-2)$ in the direction of the vector $\overrightarrow{v} = 3\widehat{\imath} - 2\widehat{\jmath}$.

8. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives for all $(x,y) \in \mathbb{R}$. Let C denote the level curve $f(x,y) = c$, for some constant c. Let (a,b) be a point on the curve C; so that $f(a,b) = c$. Assume that

$$\frac{\partial f}{\partial y}(a,b) \neq 0.$$

   Use the Chain Rule to compute the slope of the line tangent to C at the point (a,b).

9. Let $D = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$ and define $f\colon D \to \mathbb{R}$ be given by $f(x,y) = ye^{x/y}$, for all $(x,y) \in D$.

   Give the linear approximation to f at the point $(1,1)$.

10. Let $f(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$. Sketch the flow of the vector field $F(x,y) = \nabla f(x,y)$, for all $(x,y) \in \mathbb{R}^2$.