Solutions to Review Problems for Exam 3

1. For the linear system of differential equations

$$\begin{cases} \dot{x} &= y; \\ \dot{y} &= -2x - 3y, \end{cases}$$

(a) compute and sketch line–solutions, if any;

***Solution**:* Write the system in vector form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1}$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}. \tag{2}$$

The characteristic polynomial of the matrix A in (2) is

$$p_{_A}(\lambda) = \lambda^2 + 3\lambda + 2,$$

which factors into

$$p_{_A}(\lambda) = (\lambda + 2)(\lambda + 1).$$

Thus, the eigenvalues of the matrix A in (2) are

$$\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = -1.$$

Next, we find eigenvectors corresponding to the eigenvalues λ_1 and λ_2.

To find an eigenvector corresponding to $\lambda_1 = -2$, compute nontrivial solutions of the system

$$(A - \lambda_1 I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where I is the 2×2 identity matrix, or

$$\begin{cases} (0 - (-2))x + y &= 0; \\ -2x + (-3 - (-2))y &= 0, \end{cases}$$

or

$$\begin{cases} 2x + y &= 0; \\ -2x - y &= 0, \end{cases}$$

which reduces to the equation

$$2x + y = 0. \tag{3}$$

To find solutions of the equation in (3), solve for x,

$$x = -\frac{1}{2}y,$$

and set $y = -2t$, where t is a real parameter. Then, the solutions of (3) are the vectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -2t \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \text{for } t \in \mathbb{R}. \tag{4}$$

Taking $t = 1$ in (4) yields the vector

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{5}$$

which is an eigenvector of the matrix A corresponding to the eigenvalue $\lambda_1 = -2$.

To find an eigenvector corresponding to $\lambda_1 = -1$, compute nontrivial solutions of the system

$$(A - \lambda_2 I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$\begin{cases} (0-(-1))x + y &= 0; \\ -2x + (-3-(-1))y &= 0, \end{cases}$$

or

$$\begin{cases} x + y &= 0; \\ -2x - 2y &= 0, \end{cases}$$

which reduces to the equation

$$x + y = 0. \tag{6}$$

To find solutions of the equation in (6), solve for x,

$$x = -y,$$

and set $y = -t$, where t is a real parameter. Then, the solutions of (6) are the vectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -t \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \text{for } t \in \mathbb{R}. \tag{7}$$

Taking $t = 1$ in (7) yields the vector

$$\mathrm{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{8}$$

which is an eigenvector of the matrix A corresponding to the eigenvalue $\lambda_2 = -1$.

Thus, the line solutions of the system in this problem are

$$c_1 e^{-2t}\mathrm{v}_1 \quad \text{and} \quad c_2 e^{-t}\mathrm{v}_2, \quad \text{for } t \in \mathbb{R},$$

where c_1 and c_2 are non–zero constants, and v_1 and v_2 are given in (5) and (8), respectively. These are sketched in Figure 1 along with the equilibrium solution $(0, 0)$. Note that the direction of these trajectories along the line four line–solutions in the figures is towards the origin because e^{-2t} and e^{-t} decrease to 0 as t increases. □

(b) sketch the nullclines;

***Solution**:* The $\dot{x} = 0$–nullcline is the line $y = 0$ or the x–axis. On this line, the vector field associated with the system in this problem,

$$F(x, y) = \begin{pmatrix} y \\ -2x - 3y \end{pmatrix}, \tag{9}$$

is vertical. This is indicated by the vertical arrows drawn across the x–axis in Figure 1. Note that the arrows point down for positive values of x, since the field F in (9) points down for $y = 0$ and $x > 0$. To see why this is the case, note that the field F given in (9), when evaluated on the $\dot{x} = 0$–nullcline, yields

$$F(x, 0) = \begin{pmatrix} 0 \\ -2x \end{pmatrix}.$$

Thus, if $x > 0$ the field F points downwards. By the same token, the arrows point up for negative values of x.

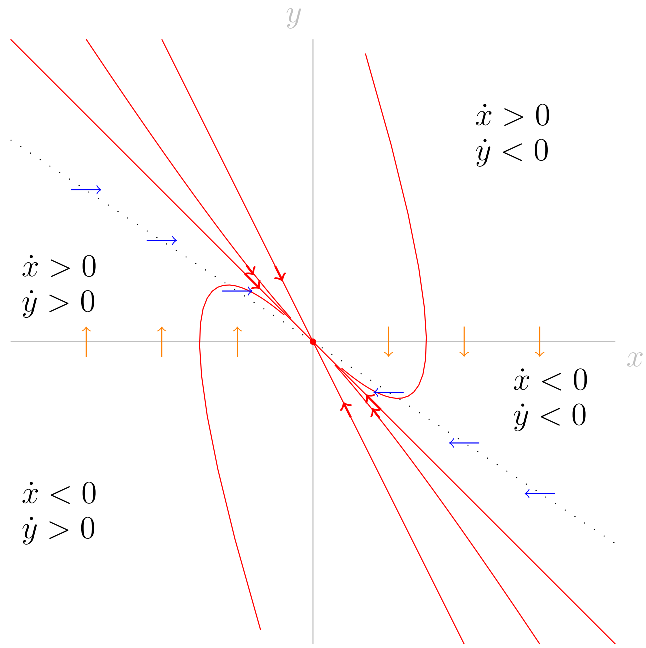

Figure 1: Sketch of phase portrait of system in Problem 1

The $\dot{y} = 0$–nullcline of the system in this problem is the line $-2x - 3y = 0$,

$$y = -\frac{2}{3}x.$$

This line is sketched as a dotted line in the sketch in Figure 1.

On the $\dot{y} = 0$–nullcline, the vector field F in (9) evaluates to

$$F(x, 0) = \begin{pmatrix} y \\ 0 \end{pmatrix}, \qquad \text{for } y = -\frac{2}{3}x. \tag{10}$$

Thus, the field is horizontal. This is indicated by the horizontal arrows on the nullcline shown in the figure. Note that, according to (10), the arrows point to the right above the x–axis ($y > 0$), and point to the left below the x–axis ($y < 0$). □

(c) sketch the phase portrait of the system;

Solution: To sketch the phase portrait of the system in this problem we use arrows on the nullclines as guide, as well as the signs of $\dot{x}$ and $\dot{y}$ determined by the differential equations, to sketch a few solution curves. Some of these trajectories are shown in Figure 1. □

(d) describe the nature of the stability (or unstability) of the origin.

***Solution**:* Since the eigenvalues of the matrix for this problem are both negative, the origin is an asymptotically stable equilibrium point; it is a sink. □

2. Let $f(x, y) = x^2 - y^2$ for all $(x, y) \in \mathbb{R}^2$, and let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field $F(x, y) = \nabla f(x, y)$, for all $(x, y) \in \mathbb{R}^2$.

(a) Sketch a contour plot for the function f.

***Solution**:* The contour plot of f consists of the graphs of the equations

$$x^2 - y^2 = c, \tag{11}$$

for various real values of c.

For instance, when $c = 0$ in (11), we get the equation

$$x^2 - y^2 = 0,$$

or

$$(x + y)(x - y) = 0,$$

which yield the lines

$$y = x \quad \text{and} \quad y = -x.$$

The graphs of these lines are shown in Figure 2.

When $c = 1$ in (11) we get the equation

$$x^2 - y^2 = 1,$$

whose graphs are hyperbolas with x–intercepts at $(\pm 1, 0)$ and asymptotes $y = \pm x$. Indeed, the graphs of the equations in (11) for $c > 0$ are hyperbolas with x–intercepts at $(\pm\sqrt{c}, 0)$ and asymptotes the lines $y = \pm x$. A couple of these hyperbolas are sketched in Figure 2.

The case $c < 0$ in (11) yields hyperbolas with y–intercepts at $(0, \pm\sqrt{-c})$ and asymptotes the lines $y = \pm x$. A couple of these hyperbolas are sketched in Figure 2. □

(b) Compute and sketch the flow of the vector field F.

***Solution**:* First, compute the gradient of f,

$$\nabla f(x, y) = \frac{\partial f}{\partial x}(x, y)\ \hat{\imath} + \frac{\partial f}{\partial y}(x, y)\ \hat{\jmath}, \quad \text{for all } (x, y) \in \mathbb{R}^2,$$

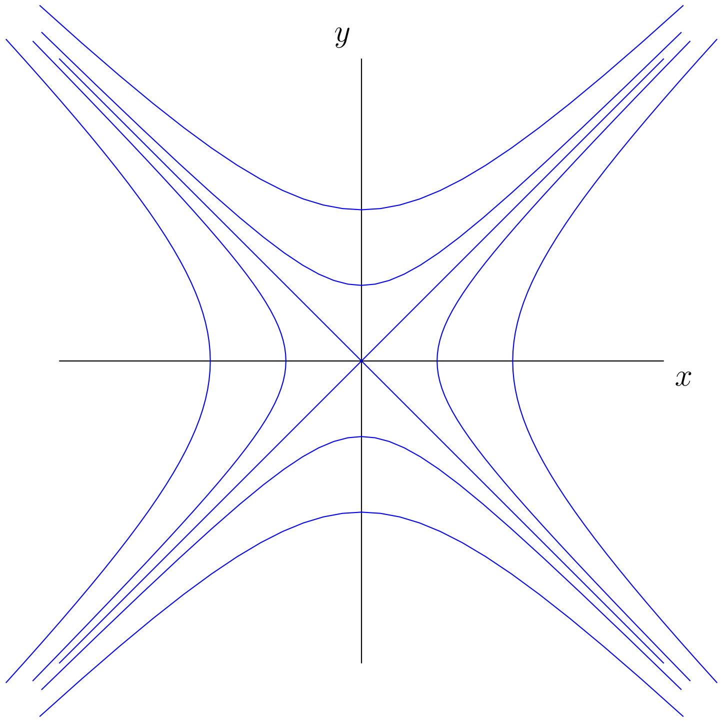

Figure 2: Contour plot of function f in Problem 2

where
$$\frac{\partial f}{\partial x}(x,y) = 2x, \quad \text{for all } (x,y) \in \mathbb{R}^2,$$
and
$$\frac{\partial f}{\partial x}(x,y) = -2y, \quad \text{for all } (x,y) \in \mathbb{R}^2.$$
Consequently,
$$\nabla f(x,y) = 2x\ \hat{i} - 2y\ \hat{j}, \quad \text{for all } (x,y) \in \mathbb{R}^2;$$
so that,
$$F(x,y) = 2x\ \hat{i} - 2y\ \hat{j}, \quad \text{for all } (x,y) \in \mathbb{R}^2. \tag{12}$$
The flow of the given vector field F in (12) are solution curves of the system of differential equations
$$\begin{cases} \dot{x} &= \ \ 2x; \\ \dot{y} &= -2y. \end{cases} \tag{13}$$

Solutions of the equations in (13) are given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-2t} \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}, \tag{14}$$

where c_1 and c_2 are arbitrary constants.

We will now proceed to sketch all types of solution curves determined by (14). These are determined by values of the parameters c_1 and c_2. For instance, when $c_1 = c_2 = 0$, (14) yields the equilibrium solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}.$$

We therefore obtain the equilibrium point $(0, 0)$ sketched in Figure 3.

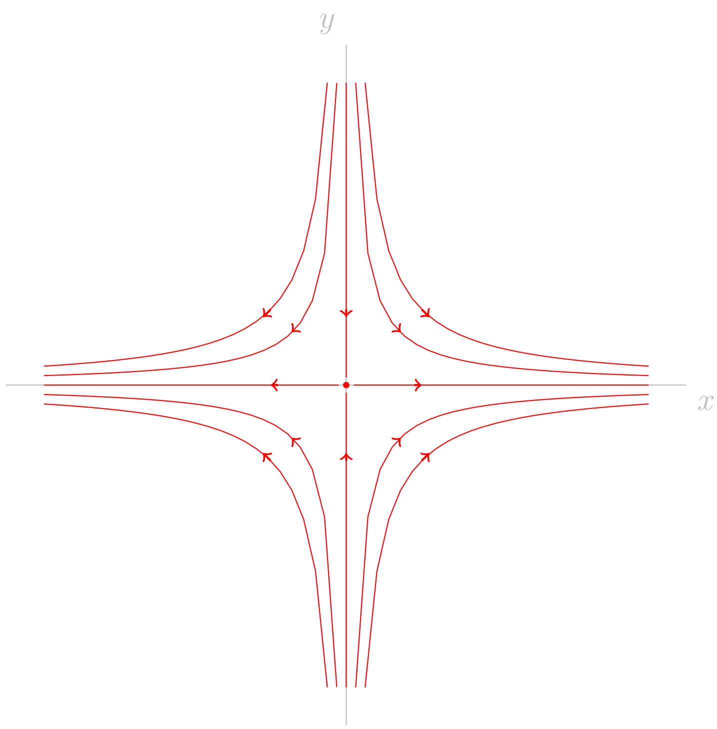

Figure 3: Sketch of Flow Field in Problem 2

Next, if $c_1 \neq 0$ and $c_2 = 0$, then the solution curve

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} \\ 0 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R},$$

will lie in the positive x–axis, if $c_1 > 0$, or in the negative x–axis if $c_1 < 0$. These two possible trajectories are shown in Figure 3. The figure also shows the trajectories tending away from the origin, as indicated by the arrows pointing towards origin. The reason for this is that, as t increases, the exponential e^{2t} increases.

Similarly, for the case $c_1 = 0$ and $c_2 \neq 0$, the solution curve

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ c_2 e^{-2t} \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

will lie in the positive y–axis, if $c_2 > 0$, or in the negative y–axis if $c_2 < 0$. In this case, the trajectories point towards the origin because the exponential function e^{-2t} decreases to 0 as t increases.

The other flow field curves correspond to the case in which $c_1 \cdot c_2 \neq 0$. To see what these curves look like, combine the two parametric equations of the curves,

$$\begin{cases} x = c_1 e^{2t}; \\ y = c_2 e^{-2t}, \end{cases} \tag{15}$$

into a single equation involving x and y by eliminating the parameter t. This can be done by multiplying the equations in (15) to get

$$xy = c_1 c_2,$$

or

$$xy = c, \tag{16}$$

where we have written c for the product $c_1 c_2$. The graphs of the equations in (16) are hyperbolas for $c \neq 0$. A few of these hyperbolas are sketched in Figure 3. Observe that all the hyperbolas in the figure have directions associated with them indicated by the arrows. The directions can be obtained from the formula for the solution curves in (14) or from the differential equations in the system in (13). For instance, in the first quadrant ($x > 0$ and $y > 0$), we get from the differential equations in (13) that $x'(t) > 0$ and $y'(t) < 0$ for all t; so that, the values of x along the trajectories increase, while the y–values decrease. Thus, the arrows point down and to the right as shown in Figure 3. □

3. Give the formula for an affine function $f\colon \mathbb{R}^2 \to \mathbb{R}$ whose graph contains the points $(1, 4, 7)$, $(4, 7, 0)$ and $(0, 4, 7)$. Sketch the graph of f.

Solution: We look for $f\colon \mathbb{R}^2 \to \mathbb{R}$ of the form

$$f(x,y) = ax + by + c, \qquad \text{for } (x,y) \in \mathbb{R}^2, \tag{17}$$

where a, b and c are to be determined so that

$$\begin{cases} f(1,4) & = & 7; \\ f(4,7) & = & 0; \\ f(0,4) & = & 7. \end{cases} \tag{18}$$

It follows from (18) and the definition of f in (17) that

$$\begin{cases} a + 4b + c & = & 7; \\ 4a + 7b + c & = & 0; \\ 4b + c & = & 7. \end{cases} \tag{19}$$

Solving for c in the third equation in (19) yields

$$c = 7 - 4b, \tag{20}$$

and substituting this in the first two equations in (19),

$$\begin{cases} a + 4b + 7 - 4b & = & 7; \\ 4a + 7b + 7 - 4b & = & 0, \end{cases}$$

or

$$\begin{cases} a & = & 0; \\ 4a + 3b & = & -7. \end{cases} \tag{21}$$

It follows from the equations in (21) that

$$\begin{cases} a & = & 0; \\ & & \\ b & = & -\dfrac{7}{3}. \end{cases} \tag{22}$$

It follows from (20) and the second equation in (22) that

$$c = \frac{49}{3}. \tag{23}$$

Combining (17), (22) and (23), we obtain that

$$f(x,y) = -\frac{7}{3}y + \frac{49}{3}, \qquad \text{for } (x,y) \in \mathbb{R}^2. \tag{24}$$

A sketch of the graph of $z = f(x,y)$, where f is given in (24), is shown in Figure 4. □

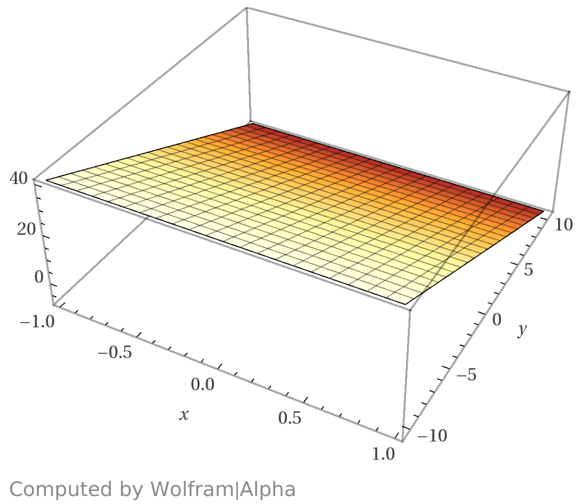

Figure 4: Sketch of plane in the system in Problem 3

4. Assume that the temperature, $T(x, y)$, at a point (x, y) in the plane is given by

$$T(x, y) = \frac{100}{1 + x^2 + y^2}, \qquad \text{for all } (x, y) \in \mathbb{R}^2.$$

(a) Sketch the contour plot for T.

***Solution**:* The contour plot of T consists of the graphs of the equations

$$T(x, y) = c, \tag{25}$$

where c is a positive constant.

The equations in (25) are equivalent to the equations

$$x^2 + y^2 = \frac{100}{c} - 1, \qquad \text{for } c > 0. \tag{26}$$

The equations in (26) describe concentric circles centered at the origin and of radius

$$\sqrt{\frac{100}{c} - 1},$$

provided that

$$\frac{100}{c} - 1 \geqslant 0,$$

or

$$0 < c \leqslant 100. \tag{27}$$

Thus, the contour plot of T consists of concentric circles around the origin as pictured in Figure 5. □

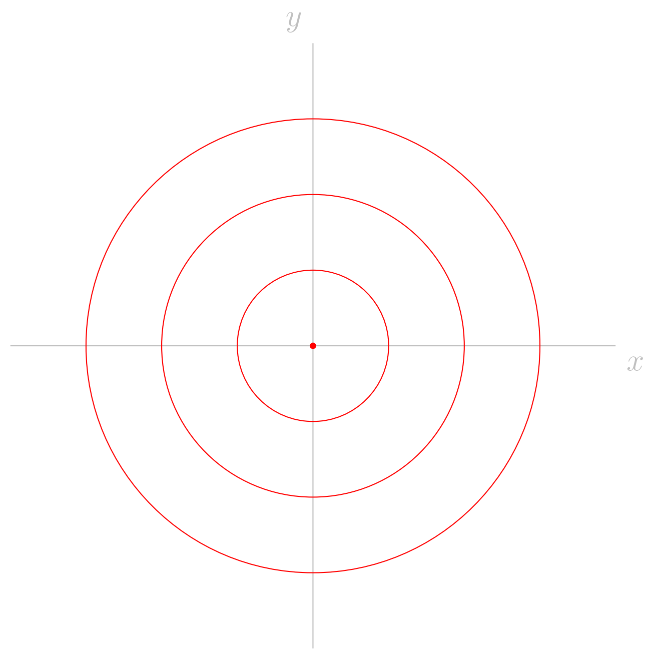

Figure 5: Contour plot of function T in Problem 4

(b) Locate the hottest point in the plane. What is the temperature at that point?

***Solution**:* The point at $(0, 0)$ in the contour plot in 5) corresponds to the level $c = 100$, which, according to (27), is the highest value that the temperature, T, can take on (see also (25)). Thus, the hottest point in the plane is the origin, and the temperature at that point is 100. □

(c) Give the direction of greatest increase in temperature at the point $(1, 1)$. What is the rate of change of temperature in that direction?

***Solution**:* The direction of maximum increase in temperature is that of the gradient of T at $(1, 1)$, or $\nabla T(1, 1)$, where

$$\nabla T(x, y) = \frac{\partial T}{\partial x}(x, y)\ \hat{i} + \frac{\partial T}{\partial y}(x, y)\ \hat{j}, \quad \text{for all } (x, y) \in \mathbb{R}^2. \tag{28}$$

Compute

$$\frac{\partial T}{\partial x}(x,y) = -\frac{200x}{(1+x^2+y^2)^2}, \qquad \text{for all } (x,y)\in\mathbb{R}^2,$$

and

$$\frac{\partial T}{\partial y}(x,y) = -\frac{200y}{(1+x^2+y^2)^2}, \qquad \text{for all } (x,y)\in\mathbb{R}^2.$$

Then,

$$\frac{\partial T}{\partial x}(1,1) = -\frac{200}{9},$$

and

$$\frac{\partial T}{\partial y}(1,1) = -\frac{200}{9}.$$

Consequently, according to (28),

$$\nabla T(1,1) = -\frac{200}{9}\ \hat{\imath} - \frac{200}{9}\ \hat{\jmath}. \qquad (29)$$

Thus, the direction of greatest increase in temperature at the point $(1,1)$ is that of the vector $\nabla T(1,1)$ given in (29), which is the same as the direction of the unit vector

$$\hat{u} = -\frac{\sqrt{2}}{2}\ \hat{\imath} - \frac{\sqrt{2}}{2}\ \hat{\imath}.$$

The rate of change of temperature in the direction of largest temperature increase is

$$D_{\hat{u}}T(1,1) = \nabla T(1,1)\cdot\hat{u} = \frac{200\sqrt{2}}{9}.$$

□

(d) A bug moves in the plane along a path given by $\sigma(t) = t\ \hat{\imath} + t^2\ \hat{\jmath}$ for $t\in\mathbb{R}$. How fast is the temperature changing when $t=1$?

***Solution**:* Apply the Chain–Rule to get

$$\frac{d}{dt}[T(\sigma(t))] = \nabla T(\sigma(t))\cdot\sigma'(t), \qquad \text{for } t\in\mathbb{R}, \qquad (30)$$

where, in this case,

$$\sigma'(t) = \hat{\imath} + 2t\hat{\jmath}, \qquad \text{for } t\in\mathbb{R}, \qquad (31)$$

Consequently, according to (30), when $t=1$,

$$\left.\frac{d}{dt}[T(\sigma(t))]\right|_{t=1} = \nabla T(1,1)\cdot\sigma'(1), \qquad (32)$$

where, in view of (31),

$$\sigma'(1) = \hat{\imath} + 2\hat{\jmath}. \tag{33}$$

It then follows from (32), (29) and (33) that

$$\begin{aligned} \frac{d}{dt}[T(\sigma(t))]\Big|_{t=1} &= \left(-\frac{200}{9}\,\hat{\imath} - \frac{200}{9}\,\hat{\jmath}\right)\cdot(\hat{\imath} + 2\hat{\jmath}) \\ &= -\frac{200}{9}(\hat{\imath} + \hat{\jmath})\cdot(\hat{\imath} + 2\hat{\jmath}) \\ &= -\frac{200}{9}(1+2); \end{aligned}$$

so that,

$$\frac{d}{dt}[T(\sigma(t))]\Big|_{t=1} = -\frac{200}{3}.$$

Thus, the rate of change of temperature along the path of the bug at $t = 1$ is $-\dfrac{200}{3}$. □

5. For the linear system of differential equations

$$\begin{cases} \dot{x} &= x + y - 1; \\ \dot{y} &= -x + y, \end{cases} \tag{34}$$

(a) sketch the nullclines and find the equilibrium points;

(b) sketch the phase portrait of the system;

(c) describe the nature of the stability (or unstability) of the equilibrium points.

***Solution**:*

(a) The $\dot{x} = 0$–nullcline is the line $x + y - 1 = 0$, or

$$x + y = 1. \tag{35}$$

On the line in (35), the direction of the vector field associated with the system in this problem,

$$F(x,y) = \begin{pmatrix} x + y - 1 \\ -x + y \end{pmatrix}, \qquad \text{for } (x,y) \in \mathbb{R}^2, \tag{36}$$

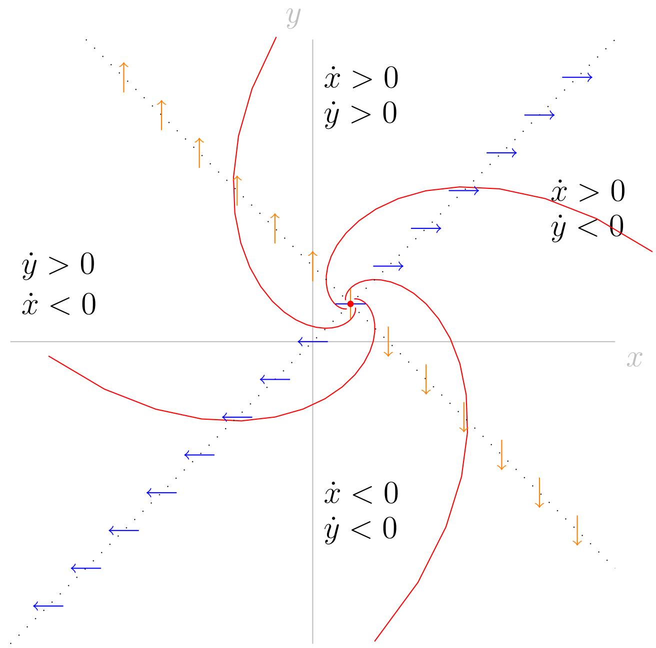

Figure 6: Sketch of phase portrait of system in Problem 5

is vertical. This is indicated by the vertical arrows on the line in (35) shown in Figure 6. The directions of the arrows are determined by the signs $\dot{y}$ determined by the second differential equation in the system in (34); these signs are shown in the sketch in Figure 6.

The $\dot{y} = 0$–nullcline is the line $-x + y = 0$, or

$$y = x. \tag{37}$$

On this line, the direction of the field F in (36) is horizontal. This is indicated by the horizontal arrows across the line in (37) shown in the sketch in Figure 6. The direction of the horizontal arrows is determined by the signs of $\dot{x}$ shown in the sketch.

The nullclines in (35) and (37 meet at the equilibrium point

$$(\overline{x}.\overline{y}) = \left(\frac{1}{2}, \frac{1}{2}\right). \tag{38}$$

This point is shown in the sketch in Figure 6.

(b) To sketch the phase portrait of the system in (34), write the system in vector form,

$$\begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{39}$$

The characteristic polynomial of the matrix A in (39) is

$$p_{_A}(\lambda) = \lambda^2 - 2\lambda + 2,$$

which has the complex roots

$$\lambda = 1 \pm i.$$

Since the real part of these eigenvalues, 1, is positive, the trajectories of the system in (34) will spiral away from the equilibeium point $(\overline{x}, \overline{y})$ in (38) in the clockwise sense according to the directions across the nollclines. A few of these spirals are shown in the sketch in Figure 6.

(c) The equilibeium point $(\overline{x}, \overline{y})$ in (38) is an unstable spiral; or a spiral source.

□

6. Sketch the flow of the linear vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, y) = (6x + 4y)\ \hat\imath - (10x + 6y)\ \hat\jmath \quad \text{for } (x, y) \in \mathbb{R}^2. \tag{40}$$

Suggestion: Sketch nullclines and determine the nature of the stability of the origin.

Solution*:* The flow of the vector field F in (40) consists of the solution curves of the system of differential equations

$$\begin{cases} \dot x = 6x + 4y; \\ \dot y = -10x - 6y. \end{cases} \tag{41}$$

The system in (41) can be written in vector form as

$$\begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 6 & 4 \\ -10 & -6 \end{pmatrix} \tag{42}$$

The characteristic polynomial of the matrix A in (42) is

$$p_{_A}(\lambda) = \lambda^2 + 4,$$

which has purely imaginary roots $\pm 2i$. Thus, the eigenvalues of the matrix A in (42) are purely imaginary. Hence, the orbits of the system in (41) are concentric ellipses centered at the origin.

To sketch the ellipses making up the phase portrait of the system in (41), we first sketch the nullclines of the system.

The $\dot{x} = 0$–nullcline is the line $6x + 4y = 0$, or

$$y = -\frac{3}{2}x. \tag{43}$$

This line is sketched in Figure 7, along with the vertical arrows across it indicating the fact that the field F in (40) is vertical on this nullcline.

The $\dot{y} = 0$–nullcline is the line $-10x - 6y = 0$, or

$$y = -\frac{5}{3}x. \tag{44}$$

On this line, the vector field F in (40) is horizontal. This is indicated in the sketch in Figure 7) by the horizontal arrows across the line.

To sketch the ellipses making up the phase portrait, use the arrows on the nullclines as guides; so that the ellipses are oriented in the clockwise sense. A few of the ellipses, as well as the equilibrium solution at the origin, are sketched in Figure 7. □

7. Let $f(x,y) = \dfrac{x+y}{1+x^2}$ for all $(x,y) \in \mathbb{R}^2$. Compute the rate of change of f at $(1,-2)$ in the direction of the vector $v = 3\hat{i} - 2\hat{j}$.

Solution: We compute

$$D_{\hat{u}} f(1,-2) = \nabla f(1,-2) \cdot \hat{u}, \tag{45}$$

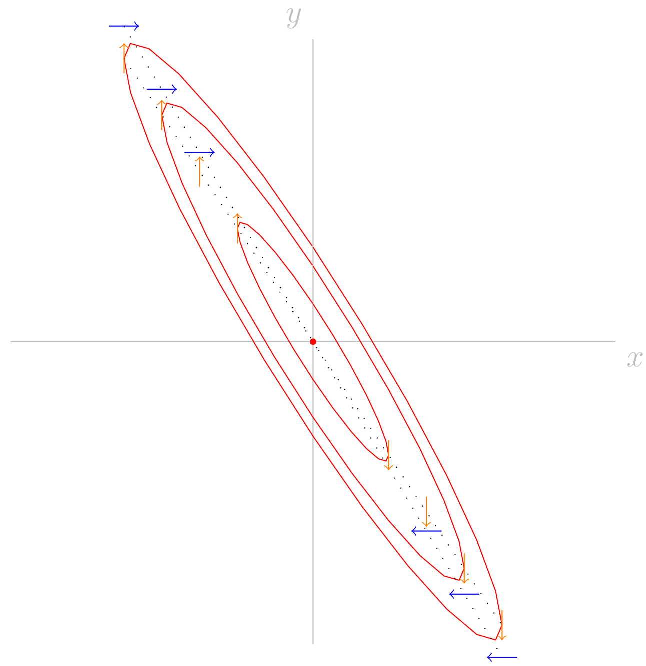

Figure 7: Sketch of flow of vector–field in Problem 6

where $\hat{u}$ is a unit vector in the direction of the vector v; so that,

$$\hat{u} = \frac{3}{\sqrt{13}}\hat{i} - \frac{2}{\sqrt{13}}\hat{j}. \tag{46}$$

The gradient of f is given by

$$\nabla f(x,y) = \frac{\partial f}{\partial x}(x,y)\ \hat{i} + \frac{\partial f}{\partial y}(x,y)\ \hat{j}, \quad \text{for all } (x,y) \in \mathbb{R}^2. \tag{47}$$

where, applying the Quotient Rule,

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \frac{\partial}{\partial x}\left[\frac{x+y}{1+x^2}\right] \\ &= \frac{(1+x^2)(1)-(x+y)(2x)}{(1+x^2)^2} \\ &= \frac{1+x^2-2x^2-2xy}{(1+x^2)^2}; \end{aligned}$$

so that,

$$\frac{\partial f}{\partial x}(x,y) = \frac{1 - x^2 - 2xy}{(1+x^2)^2}, \qquad \text{for } (x,y) \in \mathbb{R}^2. \tag{48}$$

Similarly,

$$\begin{aligned} \frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y}\left[\frac{x+y}{1+x^2}\right] \\ &= \frac{(1+x^2)(1) - (x+y)(0)}{(1+x^2)^2} \\ &= \frac{1+x^2}{(1+x^2)^2}; \end{aligned}$$

so that,

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{1+x^2}, \qquad \text{for } (x,y) \in \mathbb{R}^2. \tag{49}$$

Combining (47), (48) and (49),

$$\nabla f(x,y) = \frac{1 - x^2 - 2xy}{(1+x^2)^2}\ \hat{\imath} + \frac{1}{1+x^2}\ \hat{\jmath}, \qquad \text{for all } (x,y) \in \mathbb{R}^2. \tag{50}$$

Evaluating the expression of the gradient of f in (50) at the point $(1,-2)$ then yields

$$\nabla f(1,-2) = \hat{\imath} + \frac{1}{2}\ \hat{\jmath}. \tag{51}$$

Finally, using (45), (46) and (51), we compute

$$\begin{aligned} D_{\hat{u}} f(1,-2) &= \left(\hat{\imath} + \frac{1}{2}\ \hat{\jmath}\right) \cdot \left(\frac{3}{\sqrt{13}}\hat{\imath} - \frac{2}{\sqrt{13}}\hat{\jmath}\right) \\ &= \frac{3}{\sqrt{13}} - \frac{2}{\sqrt{13}} \\ &= \frac{1}{\sqrt{13}}. \end{aligned}$$

Hence, the rate of change of f at $(1,-2)$ in the direction of the vector $v = 3\hat{\imath} - 2\hat{\jmath}$ is $\dfrac{\sqrt{13}}{13}$. □

8. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives for all $(x, y) \in \mathbb{R}$. Let C denote the level curve $f(x, y) = c$, for some constant c. Let (a, b) be a point on the curve C; so that $f(a, b) = c$. Assume that

$$\frac{\partial f}{\partial y}(a, b) \neq 0. \tag{52}$$

Use the Chain Rule to compute the slope of the line tangent to C at the point (a, b).

***Solution**:* Let $\sigma\colon I \to \mathbb{R}^2$ be parametrization of the portion of the curve C that contains the point (a, b). Assume that $0 \in I$ and that

$$\sigma(0) = (a, b). \tag{53}$$

We are given that

$$f(\sigma(t)) = c, \qquad \text{for all } t \in I. \tag{54}$$

Take the derivative with respect to t on both sides of the expression in (54) to get

$$\frac{d}{dt}[f(\sigma(t))] = 0, \qquad \text{for all } t \in I, \tag{55}$$

since c is constant.

Applying the Chain–Rule in the left–hand side of (55) yields

$$\nabla f(\sigma(t)) \cdot \sigma'(t) = 0, \qquad \text{for all } t \in I, \tag{56}$$

where

$$\nabla f(x, y) = \frac{\partial f}{\partial x}(x, y)\ \hat{\imath} + \frac{\partial f}{\partial y}(x, y)\ \hat{\jmath}, \qquad \text{for all } (x, y) \in \mathbb{R}^2. \tag{57}$$

Writing

$$\sigma(t) = (x(t), y(t)), \qquad \text{for } t \in I, \tag{58}$$

we have that

$$\sigma'(t) = \frac{dx}{dt}\ \hat{\imath} + \frac{dy}{dt}\ \hat{\jmath}, \qquad \text{for all } t \in I. \tag{59}$$

Substituting the expression for ∇f in (57) and the expression for σ' in (59) in the left–hand side of (56), we obtain

$$\left(\frac{\partial f}{\partial x}(x, y)\ \hat{\imath} + \frac{\partial f}{\partial y}(x, y)\ \hat{\jmath}\right) \cdot \left(\frac{dx}{dt}\ \hat{\imath} + \frac{dy}{dt}\ \hat{\jmath}\right) = 0, \qquad \text{for } t \in I,$$

or

$$\frac{\partial f}{\partial x}(\sigma(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(\sigma(t))\frac{dy}{dt} = 0, \quad \text{for } t \in I. \tag{60}$$

Substituting $t = 0$ in (60) and using (53) we obtain

$$\frac{\partial f}{\partial x}(a,b)\frac{dx}{dt} + \frac{\partial f}{\partial y}(a,b)\frac{dy}{dt} = 0. \tag{61}$$

Assuming that (52) holds true, we can solve (61) for $\dfrac{dy}{dt}$ to get

$$\frac{dy}{dt} = -\frac{\dfrac{\partial f}{\partial x}(a,b)}{\dfrac{\partial f}{\partial y}(a,b)}\,\frac{dx}{dt}. \tag{62}$$

Assuming that $y'(x) = \dfrac{dy}{dx}$ exists, we can use the Chain–Rule to compute

$$\frac{dy}{dt} = y'(x)\frac{dx}{dt},$$

from which we get that

$$y'(x) = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}};$$

so that, in view of (62),

$$y'(x) = -\frac{\dfrac{\partial f}{\partial x}(a,b)}{\dfrac{\partial f}{\partial y}(a,b)}.$$

□

9. Let $D = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$ and define $f\colon D \to \mathbb{R}$ be given by $f(x,y) = ye^{x/y}$, for all $(x,y) \in D$.

   Give the linear approximation to f at the point $(1,1)$.

   ***Solution**:* The linear approximation to f at $(1,1)$ is

$$L(x,y) = f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1), \quad \text{for } (x,y) \in \mathbb{R}^2, \tag{63}$$

where, using the Chain–Rule,

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= y\frac{\partial}{\partial x}[e^{x/y}] \\
&= ye^{x/y}\frac{\partial}{\partial x}\left(\frac{x}{y}\right) \\
&= ye^{x/y}\frac{1}{y};
\end{aligned}$$

so that,

$$\frac{\partial f}{\partial x}(x,y) = e^{x/y}, \quad \text{for } (x,y) \in D. \tag{64}$$

Similarly, using the Product Rule and the Chain–Rule,

$$\begin{aligned}
\frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y}[ye^{x/y}] \\
&= e^{x/y} + ye^{x/y}\frac{\partial}{\partial y}\left(\frac{x}{y}\right) \\
&= e^{x/y} + ye^{x/y}\left(-\frac{x}{y^2}\right) \\
&= e^{x/y} - \frac{x}{y}ye^{x/y};
\end{aligned}$$

so that,

$$\frac{\partial f}{\partial y}(x,y) = \left(1 - \frac{x}{y}\right)e^{x/y}, \quad \text{for } (x,y) \in D. \tag{65}$$

Evaluating the partial derivatives of f in (64) and (65) at $(1,1)$, we obtain

$$\frac{\partial f}{\partial x}(1,1) = e \quad \text{and} \quad \frac{\partial f}{\partial y}(1,1) = 0.$$

Substituting these values into the expression for the linear approximation to f at $(1,1)$ then yields

$$L(x,y) = e + e(x-1), \quad \text{for } (x,y) \in \mathbb{R}^2,$$

or

$$L(x,y) = ex, \quad \text{for } (x,y) \in \mathbb{R}^2.$$

□

10. Let $f(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$. Sketch the flow of the vector field $F(x,y) = \nabla f(x,y)$, for all $(x,y) \in \mathbb{R}^2$.

    Solution: First, compute the gradient of f,

$$\nabla f(x,y) = \frac{\partial f}{\partial x}(x,y)\ \hat{\imath} + \frac{\partial f}{\partial y}(x,y)\ \hat{\jmath}, \quad \text{for all } (x,y) \in \mathbb{R}^2,$$

where

$$\frac{\partial f}{\partial x}(x,y) = 2x, \quad \text{for all } (x,y) \in \mathbb{R}^2,$$

and

$$\frac{\partial f}{\partial x}(x,y) = 2y, \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Consequently,

$$\nabla f(x,y) = 2x\ \hat{\imath} + 2y\ \hat{\jmath}, \quad \text{for all } (x,y) \in \mathbb{R}^2;$$

so that,

$$F(x,y) = 2x\ \hat{\imath} + 2y\ \hat{\jmath}, \quad \text{for all } (x,y) \in \mathbb{R}^2. \tag{66}$$

The flow of the given vector field F in (66) are solution curves of the system of differential equations

$$\begin{cases} \dot{x} &=& 2x; \\ \dot{y} &=& 2y. \end{cases} \tag{67}$$

Solutions of the equations in (67) are given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{2t} \end{pmatrix}, \quad \text{for } t \in \mathbb{R}, \tag{68}$$

where c_1 and c_2 are arbitrary constants.

We will now proceed to sketch all types of solution curves determined by (68). These are determined by values of the parameters c_1 and c_2. For instance, when $c_1 = c_2 = 0$, (68) yields the equilibrium solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}.$$

We therefore obtain the equilibrium point $(0,0)$ sketched in Figure 8.

Next, if $c_1 \neq 0$ and $c_2 = 0$, then the solution curve

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} \\ 0 \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

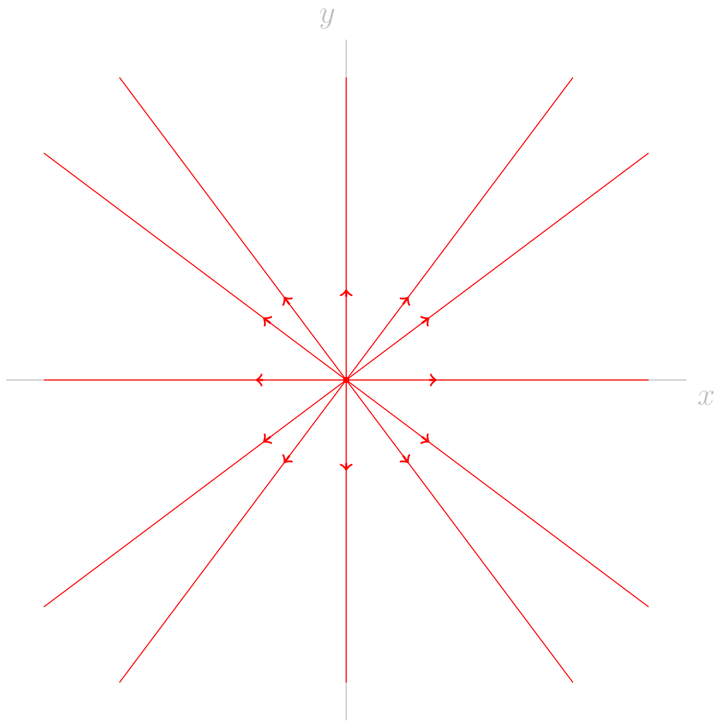

Figure 8: Sketch of Flow Field in Problem 10

will lie in the positive x–axis, if $c_1 > 0$, or in the negative x–axis if $c_1 < 0$. These two possible trajectories are shown in Figure 8. The figure also shows the trajectories tending away from the origin, as indicated by the arrows pointing away from the origin in the figure. The reason for this is that, as t increases, the exponential e^{2t} increases.

Similarly, for the case $c_1 = 0$ and $c_2 \neq 0$, the solution curve

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ c_2 e^{2t} \end{pmatrix}, \qquad \text{for } t \in \mathbb{R},$$

will lie in the positive y–axis, if $c_2 > 0$, or in the negative y–axis if $c_2 < 0$. In this case, the trajectories also point away from the origin because the exponential function e^{2t} increases to 0 as t increases.

The other flow field curves correspond to the case in which $c_1 \neq 0$ and $c_2 \neq 0$. To see what these curves look like, combine the two parametric equations of the curves,

$$\begin{cases} x = c_1 e^{2t}; \\ y = c_2 e^{2t}, \end{cases} \tag{69}$$

into a single equation involving x and y by eliminating the parameter t. This can be done by solving for e^{2t} in the first equation in (69),

$$e^{2t} = \frac{1}{c_1}x,$$

and substituting in the second equation in (69) to get to get

$$y = \frac{c_2}{c_1}x,$$

or

$$y = cx, \tag{70}$$

where we have written c for the ratio c_2/c_1. The graphs of the equations in (70) are straight lines through the origin of slope $c \neq 0$. A few of these lines are sketched in Figure 8. Observe that all the line–solutions in the figure emanate from the origin as indicated by the arrows. The directions can be obtained from the formula for the solution curves in (68) or from the differential equations in the system in (67). For instance, in the first quadrant ($x > 0$ and $y > 0$), we get from the differential equations in (67) that $x'(t) > 0$ and $y'(t) > 0$ for all t; so that, the values of x and y along the trajectories increase. Thus, the arrows point away from the origin in Figure 8. □