Assignment #18

Due on Wednesday, April 29, 2020

Read Section 8.1 on the *Definition of Convergence in Distribution* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 8.2 on the *mgf Convergence Theorem* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Do the following problems

1. Let a denote a real number and X_{a} be a discrete random variable with pmf

$$p_{X_a}(x) = \begin{cases} 1 & \text{if } x = a; \\ 0 & \text{elsewhere.} \end{cases}$$

   (a) Compute the cdf for X_{a} and sketch its graph.

   (b) Compute the mgf for X_{a} and determine $E(X_{a})$ and $\text{Var}(X_{a})$.

2. Let (X_k) denote a sequence of independent identically distributed random variables such that $X_k \sim \text{Normal}(\mu, \sigma^2)$ for every $k = 1, 2, \ldots$, and for some $\mu \in \mathbb{R}$ and $\sigma > 0$. For each $n \geqslant 1$, define

$$\overline{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

   (a) Determine the mgf, $\psi_{\overline{X}_n}(t)$, for $\overline{X}_n$, and compute $\lim_{n\to\infty} \psi_{\overline{X}_n}(t)$.

   (b) Find the limiting distribution of $\overline{X}_n$ as $n \to \infty$. (*Hint:* Compare your answer in part (a) to your answer in part (b) of problem 1.)

3. Let (X_k) and $\overline{X}_n$ be defined as in the previous problem. Define $Z_n = \dfrac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$ for all $n \geqslant 1$.

   (a) Determine the mgf, $\psi_{Z_n}(t)$, for Z_n, and compute $\lim_{n\to\infty} \psi_{Z_n}(t)$.

   (b) Find the limiting distribution of Z_n as $n \to \infty$.

4. Let (Y_n) be a sequence of discrete random variables having pmfs

$$p_{Y_n}(y) = \begin{cases} 1 & \text{if } y = n, \\ 0 & \text{elsewhere.} \end{cases}$$

   Compute the mgf of Y_n for each $n = 1, 2, 3, \ldots$

   Does $\lim_{n\to\infty} \psi_{Y_n}(t)$ exist for any t in an open interval around 0?

   Does the sequence (Y_n) have a limiting distribution? Justify your answer.

5. Let $q = 0.95$ denote the probability that a patient receiving a treatment lives for at least 5 years after the treatment.

   (a) If we observe 60 people from a group that received the treatment, what is the probability that at least 56 of them live 5 years or more? State the assumptions that you made in your solution.

   (b) Find and approximation to the result of part (a) using the Poisson distribution. Explain the reasoning leading to your answer.