Solutions to Review Problems for Exam 1

1. There are 5 red chips and 3 blue chips in a bowl. The red chips are numbered $1, 2, 3, 4, 5$ respectively, and the blue chips are numbered $1, 2, 3$ respectively. If two chips are to be drawn at random and without replacement, find the probability that these chips are have either the same number or the same color.

   ***Solution**:* The sample space for drawing two chips at random out of a bowl containing 8 chips consists of
$$\frac{8 \times 7}{2} = 28 \tag{1}$$
   pairs of chips. The argument behind the calculation in (1) is as follows: There are 8 choices for the first draw. Because the sampling is without replacement, there are 7 choices for the second draw. Since the chips are drawn two at a time, the order of the draw does not matter; thus, we need to divide by 2 because there 2 ways in which the chips in the pair can be ordered. That is why we divided by 2 in the expression in (1).

   The assumption of randomness in the draws implies that all the elements of the sample space have the same likelihood of $1/28$.

   Let R denote the event that the two chips are red. Since there are 5 red chips in the bowl, there are
$$\frac{5 \times 4}{2} = 10$$
   pairs of red chips in the sample space. Therefore, by the equal likelihood assumption,
$$\Pr(R) = \frac{10}{28} = \frac{5}{14}. \tag{2}$$
   Let B denote the event that both chips are blue. Then, B consists of
$$\frac{3 \times 2}{2} = 3$$
   pairs of blue chips in the sample space. Consequently,
$$\Pr(B) = \frac{3}{28}. \tag{3}$$
   Observe that the events R and B are disjoint; thus, by the finite additivity property,
$$\Pr(R \cup B) = \Pr(R) + \Pr(B) = \frac{13}{28}, \tag{4}$$

where we have used the results in (2) and (3).

Note that $R \cup B$ is the event that both chips are of the same color.

Let N denote the event that both chips show the same number. Then, N consists of exactly three outcomes in the sample space; accordingly,

$$\Pr(N) = \frac{3}{28}. \tag{5}$$

Finally, since $R \cup B$ and N are disjoint, the probability that the chips are have either the same number or the same color is

$$\Pr(R \cup B \cup N) = \Pr(R \cup B) + \Pr(N) = \frac{13}{28} + \frac{3}{28} = \frac{4}{7},$$

where we have used the finite additivity property and (4) and (5). □

2. A person has purchased 10 of 1,000 tickets sold in a certain raffle. To determine the five prize winners, 5 tickets are drawn at random and without replacement. Compute the probability that this person will win at least one prize.

***Solution**:* Let N denote the event that the person will not win any prize. Then, N^c is the event that the person will win at least one prize. Thus, we will compute $\Pr(N)$ to get

$$\Pr(N^c) = 1 - \Pr(N), \tag{6}$$

by the complement rule of probability.

Let N_1 denote the event that the person does not win in the first draw. Then,

$$\Pr(N_1) = \frac{990}{1000}, \tag{7}$$

since there are 990 ways of not picking one of the 10 tickets that the person bought.

Letting N_2 denote the event of not winning in the second draw. Then, by the multiplication rule

$$\Pr(N_1 \cap N_2) = \Pr(N_1) \cdot \Pr(N_1 \mid N_1),$$

where

$$\Pr(N_1 \mid N_1) = \frac{989}{999},$$

since there are 989 ways of picking a non–wining ticket in the second draw once a non–winning ticket has been drawn in the first draw. Consequently, using (7),

$$\Pr(N_1 \cap N_2) = \frac{990}{1000} \cdot \frac{989}{999}.$$

Continuing in this fashion, letting N_k denote the event of not drawing the wining ticket in the k^{th} draw, we get that

$$\Pr(N_1 \cap N_2 \cap N_3 \cap N_4 \cap N_5) = \frac{990}{1000} \cdot \frac{989}{999} \cdot \frac{988}{998} \cdot \frac{987}{997} \cdot \frac{986}{996}. \tag{8}$$

Observe that $N = N_1 \cap N_2 \cap N_3 \cap N_4 \cap N_5$. It then follows from (8) that

$$\begin{aligned} \Pr(N) &= \frac{(990)(989)(988)(987)(986)}{(1000)(999)(998)(997)(996)} \\ &= \frac{435841667261}{458349513900}; \end{aligned}$$

so that,

$$\Pr(N) \approx 0.9509. \tag{9}$$

Finally, combining the results in (6) and (9), we get that the probability of the person winning at least one of the prizes is

$$\Pr(N^c) \approx 0.0491,$$

or about 4.91%. □

3. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let E_1, E_2 and E_3 be mutually disjoint events in $\mathcal{B}$. Find $\Pr[(E_1 \cup E_2) \cap E_3]$ and $\Pr(E_1^c \cup E_2^c)$.

   ***Solution**:* Since E_1, E_2 and E_3 are mutually disjoint events, it follows that $(E_1 \cup E_2) \cap E_3 = \emptyset$; so that

   $$\Pr[(E_1 \cup E_2) \cap E_3] = 0.$$

   Next, use De Morgan's law to compute

   $$\begin{aligned} \Pr(E_1^c \cup E_2^c) &= \Pr([E_1 \cap E_2]^c) \\ &= \Pr(\emptyset^c) \\ &= \Pr(\mathcal{C}) \\ &= 1. \end{aligned}$$

□

4. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let A and B events in $\mathcal{B}$. Show that
$$\Pr(A \cap B) \leq \Pr(A) \leq \Pr(A \cup B) \leq \Pr(A) + \Pr(B). \tag{10}$$

***Solution**:* Since $A \cap B \subseteq A$, it follows that
$$\Pr(A \cap B) \leqslant \Pr(A), \tag{11}$$
by the monotonicity property of probability.

Similarly, since $A \subseteq A \cup B$, we get that
$$\Pr(A) \leqslant \Pr(A \cup B). \tag{12}$$

Next, use the inclusion–exclusion property of probability,
$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B),$$
and fact that that
$$\Pr(A \cap B) \geqslant 0,$$
by the second Kolmogorov axiom of probability, to obtain that
$$\Pr(A \cup B) \leqslant \Pr(A) + \Pr(B). \tag{13}$$

Finally, combine (11), (12) and (13) to obtain (10). □

5. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let E_1, E_2 and E_3 be mutually independent events in $\mathcal{B}$ with probabilities $\dfrac{1}{2}$, $\dfrac{1}{3}$ and $\dfrac{1}{4}$, respectively. Compute the exact value of $\Pr(E_1 \cup E_2 \cup E_3)$.

***Solution**:* First, use De Morgan's law to compute
$$\Pr[(E_1 \cup E_2 \cup E_3)^c] \;=\; \Pr(E_1^c \cap E_2^c \cap E_3^c) \tag{14}$$

Then, since E_1, E_2 and E_3 are mutually independent events, it follows from (14) that
$$\Pr[(E_1 \cup E_2 \cup E_3)^c] \;=\; \Pr(E_1^c) \cdot \Pr(E_2^c) \cdot \Pr(E_3^c),$$

so that

$$\begin{aligned}\Pr[(E_1 \cup E_2 \cup E_3)^c] &= (1 - \Pr(E_1))(1 - \Pr(E_2))(1 - \Pr(E_3)) \\ &= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \\ &= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4},\end{aligned}$$

so that

$$\Pr[(E_1 \cup E_2 \cup E_3)^c] = \frac{1}{4}. \tag{15}$$

It then follows from (15) that

$$\Pr(E_1 \cup E_2 \cup E_3) = 1 - \Pr[(E_1 \cup E_2 \cup E_3)^c] = \frac{3}{4}.$$

□

6. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let E_1, E_2 and E_3 be mutually independent events in $\mathcal{B}$ with $\Pr(E_1) = \Pr(E_2) = \Pr(E_3) = \dfrac{1}{4}$. Compute $\Pr[(E_1^c \cap E_2^c) \cup E_3]$.

   Solution: First, use De Morgan's law to compute

$$\Pr[((E_1^c \cap E_2^c) \cup E_3)^c] = \Pr[(E_1^c \cap E_2^c)^c \cap E_3^c] \tag{16}$$

   Next, use the assumption that E_1, E_2 and E_3 are mutually independent events to obtain from (16) that

$$\Pr[((E_1^c \cap E_2^c) \cup E_3)^c] = \Pr[(E_1^c \cap E_2^c)^c] \cdot \Pr[E_3^c], \tag{17}$$

   where

$$\Pr[E_3^c] = 1 - \Pr(E_3) = \frac{3}{4}, \tag{18}$$

   and

$$\begin{aligned}\Pr[(E_1^c \cap E_2^c)^c] &= 1 - \Pr[E_1^c \cap E_2^c] \\ &= 1 - \Pr[E_1^c] \cdot \Pr[E_2^c],\end{aligned} \tag{19}$$

   by the independence of E_1 and E_2.

It follows from the calculations in (19) that

$$\begin{aligned}
\Pr[(E_1^c \cap E_2^c)^c] &= 1 - (1 - \Pr[E_1])(1 - \Pr[E_2]) \\
&= 1 - \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{4}\right) \\
&= 1 - \frac{3}{4} \cdot \frac{3}{4} \\
&= \frac{7}{16}
\end{aligned} \tag{20}$$

Substitute (18) and the result of the calculations in (20) into (17) to obtain

$$\Pr[((E_1^c \cap E_2^c) \cup E_3)^c] = \frac{7}{16} \cdot \frac{3}{4} = \frac{21}{64}. \tag{21}$$

Finally, use the result in (21) to compute

$$\begin{aligned}
\Pr[(E_1^c \cap E_2^c) \cup E_3^c] &= 1 - \Pr[((E_1^c \cap E_2^c) \cup E_3)^c] \\
&= 1 - \frac{21}{64} \\
&= \frac{43}{64}.
\end{aligned}$$

□

7. A bowl contains 5 chips of the same size and shape. One the chips is red and the rest are blue. Draw chips from the bowl at random, one at a time and without replacement, until the red chip is drawn.

   (a) Describe the sample space of this experiment.

   ***Solution**:* Denoting the red chip by R and any of the blue chips by B, we have that the sample space for this experiment is

$$\mathcal{C} = \{R, BR, BBR, BBBR, BBBBR\}.$$

□

(b) Define the probability function for this experiment. Justify your answer.

***Solution**:* Since we are assuming that the chips are drawn at random and without replacement , we have that

$$\Pr(R) = \frac{1}{5};$$

$$\Pr(BR) = \frac{4}{5}\cdot\frac{1}{4} = \frac{1}{5};$$

$$\Pr(BBR) = \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{1}{3} = \frac{1}{5};$$

$$\Pr(BBBR) = \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2} = \frac{1}{5};$$

and

$$\Pr(BBBBR) = \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{5}.$$

Thus, we conclude that

$$\Pr(c) = \frac{1}{5}, \quad \text{for all } c \in \mathcal{C}.$$

□

(c) Compute the probability that at least two draws will be needed to get the red chip.

***Solution**:* The event, E, that at least two draws will be needed to get the red chip, is the complement of the set $\{R\}$. Thus, $E = \{R\}^c$ and therefore

$$\Pr(E) = 1 - \Pr(\{R\}) = 1 - \frac{1}{5} = \frac{4}{5}.$$

□

8. Dreamboat cars are produced at three different factories A, B and C. Factory A produces 20 percent of the total output of Dreamboats, B produces 50 percent, and C produces 30 percent. However, 5 percent of the cars produced at A are lemons, 2 percent of those produced at B are lemons, and 10 percent of those produced at C are lemons. If you buy a Dreamboat and it turns out to be lemon, what is the probability that it was produced at factory A?

***Solution**:* Let A denote the event that the car was produced in Factory A, B the event the car was made in Factory B, and C the event the car was made in Factory C. We then have that

$$\Pr(A) = 0.20, \quad \Pr(B) = 0.50 \quad \text{and} \quad \Pr(C) = 0.30.$$

Let L denote the event that a given car is a lemon. We are then given the conditional probabilities

$$\Pr(L \mid A) = 0.05, \quad \Pr(L \mid B) = 0.02, \quad \text{and} \quad \Pr(L \mid C) = 0.10.$$

We want to compute $\Pr(A \mid L)$,

$$\Pr(A \mid L) = \frac{\Pr(A \cap L)}{\Pr(L)},$$

where

$$\Pr(A \cap L) = \Pr(A) \cdot \Pr(L \mid A) = (0.20) \cdot (0.05) = 0.01,$$

and

$$\begin{aligned} \Pr(L) &= \Pr(A) \cdot \Pr(L \mid A) + \Pr(B) \cdot \Pr(L \mid B) + \Pr(C) \cdot \Pr(L \mid C) \\ &= (0.20) \cdot (0.05) + (0.50) \cdot (0.02) + (0.30) \cdot (0.10) \\ &= 0.01 + 0.01 + 0.03 \\ &= 0.05. \end{aligned}$$

Hence,

$$\Pr(A \mid L) = \frac{0.01}{0.05} = \frac{1}{5},$$

or 20%. □

9. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let A and B events in $\mathcal{B}$. Given that $\Pr(A) = 1/3$, $\Pr(B) = 1/5$ and $\Pr(A \mid B) + \Pr(B \mid A) = 2/3$, compute $\Pr(A^c \cup B^c)$.

   ***Solution**:* Assume that

$$\Pr(A) = \frac{1}{3}, \qquad \Pr(B) = \frac{1}{5}, \tag{22}$$

and

$$\Pr(A \mid B) + \Pr(B \mid A) = \frac{2}{3}. \tag{23}$$

First, use De Morgan's Law and the Rule of Complements to compute

$$\begin{aligned} \Pr(A^c \cup B^c) &= \Pr((A \cap B)^c) \\ &= 1 - \Pr(A \cap B); \end{aligned}$$

so that
$$\Pr(A^c \cup B^c) = 1 - \Pr(A) \cdot \Pr(B \mid A). \tag{24}$$
Thus, we need to compute $\Pr(B \mid A)$. To do so, first use
$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$
to obtain
$$\Pr(A \mid B) = \frac{\Pr(A) \cdot \Pr(B \mid A)}{\Pr(B)},$$
or
$$\Pr(A \mid B) = \frac{5}{3} \cdot \Pr(B \mid A), \tag{25}$$
in view of (22). Next, combine (25) and (23) to obtain
$$\frac{5}{3} \cdot \Pr(B \mid A) + \Pr(B \mid A) = \frac{2}{3},$$
from which we get
$$\Pr(B \mid A) = \frac{1}{4}.$$
Using this value in (24) and the value of $\Pr(A)$ in (22) we obtain that
$$\Pr(A^c \cup B^c) = 1 - \frac{1}{3} \cdot \frac{1}{4},$$
from which we get that
$$\Pr(A^c \cup B^c) = \frac{11}{12}.$$
□

10. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and let A and B independent events in $\mathcal{B}$ with $\Pr(B) > 0$. Given that $\Pr(A) = 1/3$, compute $\Pr(A \cup B^c \mid B)$.

    ***Solution**:* Use the definition of conditional probability to compute
$$\Pr(A \cup B^c \mid B) \;\; = \;\; \frac{\Pr((A \cup B^c) \cap B)}{\Pr(B)}, \tag{26}$$
    where, by the distributive property,
$$(A \cup B^c) \cap B = (A \cap B) \cup (B^c \cap B) = (A \cap B) \cup \emptyset = A \cap B;$$
    so that,
$$\Pr((A \cup B^c) \cap B) = \Pr(A \cap B),$$

and, using the assumption of independence of A and B,

$$\Pr((A \cup B^c) \cap B) = \Pr(A) \cdot \Pr(B).$$

Consequently, in view of (26),

$$\Pr(A \cup B^c \mid B) = \Pr(A) = \frac{1}{3}.$$

$\square$