Assignment #5

Due on Friday, March 6, 2020

Section 3.1 on *Modeling Traffic Flow* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`.

Read Section 3.2 on *Analysis of the Traffic Flow Equation* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

Background and Definitions

The Lighthill–Whitman–Richards Model. The following model for traffic flow on a one–lane road was proposed by Lighthill and Whitman in 1955 and by Richards in 1956.

$$\begin{cases} \dfrac{\partial u}{\partial t} + g'(u)\dfrac{\partial u}{\partial x} = 0; \\ u(x,0) = f(x), \end{cases} \tag{1}$$

where $g(u) = u(1-u)$, and f is the initial traffic density.

Do the following problems

1. Suppose the initial traffic density, f, in the initial value problem (IVP) in (1) is given by

$$f(x) = \begin{cases} 1, & \text{if } x < -1; \\ \dfrac{1}{2}(1-x), & \text{if } -1 \leqslant x < 1; \\ 0, & \text{if } x \geqslant 1. \end{cases}$$

   (a) Sketch the characteristic curves of the partial differential equation in (1).

   (b) Explain how the initial value problem in (1) can be solved in this case, and give a formula for $u(x,t)$.

2. **Traffic Flow at a Red Light**. Let the initial traffic density in the IVP (1) be given by $f(x) = 1$ for $x \leqslant 0$ and $f(x) = 0$ for $x > 0$.

   (a) Explain why, for this initial condition, the IVP (1) models the situation at a traffic light before the light turns green.

   (b) Sketch the characteristic curves of the partial differential equation in (1).

   (c) Explain why a shock wave solution does not develop at $t = 0$.

3. **Traffic Flow at a Red Light, Continued**. Let the initial traffic density be as given in problem 2.

   Look for a solution to the IVP (1) of the form

   $$u(x,t) = \varphi\left(\frac{x}{t}\right), \quad \text{for } -t < x < t, \quad \text{and} \quad t > 0,$$

   where φ is a differentiable function of a single variable.

   Suggestion: Introduce a new variable $\eta = \dfrac{x}{t}$, and compute $\dfrac{d\varphi}{d\eta}$.

4. **Traffic Flow at a Red Light, Continued**. Let the initial traffic density in IVP (1) by given by f_ε defined by

   $$f_\varepsilon(x) = \begin{cases} 1, & \text{if } x \leqslant 0; \\ 1 - \dfrac{x}{\varepsilon}, & \text{if } 0 < x \leqslant \varepsilon; \\ 0, & \text{if } x > \varepsilon, \end{cases}$$

   for $\varepsilon > 0$.

   (a) Sketch the characteristic curves of the partial differential equation in (1).

   (b) Explain how the initial value problem can be solved in this case, for each $\varepsilon > 0$. Denote the solution by u_ε and give a formula for computing $u_\varepsilon(x,t)$, for $x \in \mathbf{R}$ and $t > 0$, for any given value of $\varepsilon > 0$.

5. **Traffic Flow at a Red Light, Continued**. Let f_ε be as defined in Problem 4, and u_ε be as computed in Problem 4.

   (a) Compute $\lim\limits_{\varepsilon \to 0} u_\varepsilon(x,t)$.

   (b) Explain why the limit computed in part (a) gives a solution of the traffic flow equation for traffic at a red light before it turns green.