Assignment #7

Due on Monday, April 6, 2020

Read Section 4.1.2 on *Discrete Random Variables* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

Read Section 4.1.3 on *The Binomial Distribution* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

Read Section 4.1.4 on *Expected Value* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

Do the following problems

1. Given a discrete random variable X with a finite number of possible values

$$x_1, x_2, x_3, \ldots, x_N,$$

the expected value of X is defined to be the sum $E(X) = \sum_{i=1}^{N} x_i P[X = x_i]$. Use this formula to compute the expected value of the numbers appearing on the top face of a fair die. Explain the meaning of this number.

2. Consider the following random experiment: Assume you have a fair die and you toss it until you get a six on the top face, and then you stop. Let X denote the number of tosses you make until you stop.

   (a) Explain why X is a discrete random variable. What are the possible value for X?

   (b) For each value x of X, compute $P[X = x]$; this is called the *probability mass function,* or pmf, of the random variable X.

3. Given a discrete random variable X with an infinite number of possible values

$$x_1, x_2, x_3, \ldots$$

the expected value of X is defined to be the infinite series

$$E(X) = \sum_{i=1}^{\infty} x_i P[X = x_i].$$

Use this formula to compute the expected value random variable X of the previous problem; that is, X is the number of times you need to toss a fair die until you get a six on the top face.

4. Let $M(t)$ denote number of bacteria in a colony of initial size N_o which develop mutations in the time interval $[0, t]$. It was shown in the lectures that if there are no mutations at time $t = 0$, and if $M(t)$ follows the assumptions of a Poisson process, then the probability of no mutations in the time interval $[0, t]$ is given by

$$P_0(t) = P[M(t) = 0] = e^{-\lambda t}$$

where $\lambda > 0$ is the average number of mutations per unit time, or the *mutation rate.*

Let $T > 0$ denote the time at which the first mutation occurs.

(a) Explain why T is a random variable. Observe that it is a *continuous* random variable.

(b) For any $t > 0$, explain why the statement

$$P[T > t] = P[M(t) = 0]$$

is true, and use it to compute

$$F(t) = P[T \leq t].$$

The function $F(t)$, usually denoted by $F_T(t)$, is called the *cumulative distribution function,* or cdf, of the random variable T.

(c) Compute the derivative $f(t) = F'(t)$ of the cdf F obtained in the previous part.

The function $f(t)$, usually denoted by $f_T(t)$, is called the *probability density function,* or pdf, of the random variable T.

5. Given a continuous random variable X with pdf f_X, the *expected value* of X is defined to be

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Use this formula to compute the expected value of the T, where T is the random variable defined in the previous problem; that is, $T > 0$ is he time at which the first mutation occurs for a bacterial colony exposed to a virus at time $t = 0$, assuming that there are no mutations at that time. How does this value relate to the average mutation rate λ?