Homework 10: The Fundamental Group and Contractibility

"It now lately sometimes seemed like a kind of black miracle to me," says Hal, "that people could actually care deeply about a subject or pursuit, and could go on caring this way for years on end. Could dedicate their entire lives to it. It seemed admirable and at the same time pathetic. We are all dying to give our lives away to something, maybe. God or Satan, politics or grammar, topology or philately - the object seemed incidental to this will to give oneself away, utterly."

-David Foster Wallace, Infinite Jest

1. Let f be a path from x to y and let $u_f : \pi_1(X, x) \to \pi_1(X, y)$ be the isomorphism $u_f([g]) = [\bar{f} * g * f]$. Prove that u_f is independent of the particular path f iff $\pi_1(X, x)$ is abelian (i.e. commutative).

2. A space X is said to be *contractible* to a point $p \in X$ with p held fixed if there is a homotopy $F : X \times I \to X$ such that $F(x, 0) = p$, $F(x, 1) = x$ for all $x \in X$, and $F(p, t) = p$ for all $t \in I$.

For each $n \in \mathbb{N}$, let $X_n = \{\frac{1}{n}\} \times I$. Let $X = (I \times \{0\}) \cup (\{0\} \times I) \cup \bigcup_{n \in \mathbb{N}} X_n$ as a subspace of $\mathbb{R}^2$ with the usual topology. Prove that X is not contractible to $(0, 1)$ with $(0, 1)$ held fixed. (Hint: you may want to use ideas from the solution to problem 5 on Homework 6 and/or the proof that the Flea and the Comb space is not path connected)

3. Let $X = U \cup V$ where U and V are open in X. Let $a \in U \cap V$, and suppose that U and V are both simply connected. Prove that if $U \cap V$ is path connected then X is simply connected. (Hint: use Homework 6 problem 2)

4. Let $p \in \mathbb{R}^{n+1}$. Prove that $\mathbb{R}^{n+1} - \{p\}$ is homotopy equivalent to $S^n = \{x \in \mathbb{R}^{n+1} | \|x\| = 1\}$.

5. Prove that S^2 is simply connected.