Homework 11: Covering Spaces

"When he was a teenager the rigid drills of schooling had made him think that mathematics was just felicity with a particular kind of minutiae, knowing things, a sort of high-grade coin collecting. You learned relations and theorems and put them together. Only slowly did he glimpse the soaring structures above each discipline. Great spans joined the vistas of topology to the infinitesimal intricacies of differentials, or the plodding styles of number theory to the shifting sands of group analysis. Only then did he see mathematics as a landscape, a territory of the mind to rove and scout." -Gregory Benford, in Foundation's Fear

1. Let $p : \widetilde{X} \to X$ be a covering map with $\widetilde{X}$ path connected. Let A be a subset of X such that the inclusion induced homomorphism $i_* : \pi_1(A, a) \to \pi_1(X, a)$ is onto. Let x_1 and x_2 be elements of $p^{-1}(a)$. Prove that there is a path in $p^{-1}(A)$ from x_1 to x_2.

2. Suppose that $p : \widetilde{X} \to X$ is a covering map and X is path connected. Let x and y be points in X. Prove that there is a bijection from $p^{-1}(x)$ to $p^{-1}(y)$. If $p^{-1}(x)$ has finitely many points, say n, we say p is an n-fold covering.

3. A continuous map $f : X \to Y$ is called a *local homeomorphism* if for every $x \in X$ there is an open set V containing x such that $f(V)$ is open in Y, and $f|V : V \to f(V)$ is a homeomorphism.

Let X be a compact non-empty space, and let Y be Hausdorff, and let $f : X \to Y$ be a local homeomorphism.

a) Prove that for any point $y \in Y$ the set $f^{-1}(\{y\})$ is finite.

b) If Y is connected prove that f is onto.

4. Let X be compact and Hausdorff, and Y be Hausdorff, and let $f : X \to Y$ be a local homeomorphism which is onto. Prove that f is a covering map.