Homework 4: Quotient Topology

"It is perhaps the area of mathematics in which the greatest number of entirely new ideas has appeared, many of which have had unexpected repercussions in theories which seem very remote from it. It forms an imposing edifice, constantly under renovation, and of such complexity that very few specialists are capable of encompassing all of it." - Jean Dieudonné

1. Let $f : X \to Y$ be a continuous function between topological spaces X and Y. Let $\sim$ be some equivalence relation on X such that if $p \sim q$ then $f(p) = f(q)$, (note this is not iff). Prove that there exists a continuous function $g : (X/\sim) \to Y$ such that $f = g\pi$.

2. Let (X, Ω_X) and (Y, Ω_Y) be topological spaces, and let $f : (X, \Omega_X) \to (Y, \Omega_Y)$ be a continuous surjection. Suppose that for every topological space (Z, Ω_Z) and every function $g : (Y, \Omega_Y) \to (Z, \Omega_Z)$, if $g \circ f : (X, \Omega_X) \to (Z, \Omega_Z)$ is continuous, then g is continuous. Prove that f is a quotient map.

3. Let $X = [0, 1] \times [0, 1]$ considered as a subspace of $\mathbb{R}^2$ with the usual topology. Define $\sim$ on X by declaring $(s, t) \sim (p, q)$ iff $t = q > 0$ or $(s, t) = (p, q)$.
a) Draw a picture of $X/\sim$.
b) Is $X/\sim$ metrizable? Prove all claims.

4. Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous surjection. Let (Y, τ_f) denote Y with the quotient topology with respect to f.
a) Suppose that f is a closed map, prove that $(Y, \tau_Y) \cong (Y, \tau_f)$.
b) Give an example of a continuous surjection $f : (X, \tau_X) \to (Y, \tau_Y)$, where (Y, τ_Y) is not homeomorphic to (Y, τ_f). Prove all claims.

5. Consider $\mathbb{R}$ with the usual topology. Define an equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ iff $x - y$ is a rational number. Prove that $\mathbb{R}/\sim$ has the indiscrete topology.