Homework 5: Product spaces

"According to Woody Allen, fake rubber inkblots were originally 11 feet in diameter and fooled nobody. Later, however, a Swiss physicist 'proved that an object of a particular size could be reduced in size simply by making it smaller, a discovery that revolutionized the fake inkblot business. This little tale could be interpreted as a parody of topology, a subject whose insights at first look do seem a little obvious....There is much more to topology than fake rubber inkblots." - John Allen Paulos

1. Let $X = \mathbb{R}^{\mathbb{N}} = \{(x_1, x_2, ...)|x_i \in \mathbb{R}\}$ with the product topology. Let S denote the set of all points $(x_1, x_2, ...) \in X$ such that $x_i \neq 0$ for only finitely many values of i. Find $\mathrm{Cl}(S)$ and prove your answer.

2. The *graph* of a function $f : X \to Y$ is the set of points in $X \times Y$ of the form $(x, f(x))$ for each $x \in X$. Show that if $f : X \to Y$ is a continuous function between topological spaces, then the graph of f with the subspace topology, is homeomorphic to X.

3. Let $X \times Y$ be the product of two topological spaces. Let A be a subset of X and let B be a subset of Y. Determine whether $\mathrm{Cl}(A \times B) = \mathrm{Cl}(A) \times \mathrm{Cl}(B)$. Prove your answer

4. For each $i \in I$, let X_i be a topological space, and let $\prod_{i \in I} X_i$ denote the product space. Let $A_i \subseteq X_i$ for each i. Give a proof or counterexample to the following statements

a) $\prod_{i \in I} \mathrm{Int}(A_i) \subseteq \mathrm{Int}(\prod_{i \in I} A_i)$

b) $\mathrm{Int}(\prod_{i \in I} A_i) \subseteq \prod_{i \in I} \mathrm{Int}(A_i)$

5. Consider the infinite collection of topological spaces (X_1, τ_1), (X_2, τ_2),...,$(X_n \tau_n)$,.... Let $Y = \prod_{i \in \mathbb{N}} X_i$, with the topology $\omega = \{$unions of sets of the form $\prod_{i \in \mathbb{N}} U_i$ with each $U_i \in \tau_i\}$. Let A be a topological space, and suppose that for each i, the function $f_i : A \to X_i$ is continuous. Let $h : A \to Y$ be given by $h(a) = f$ such that $f(i) = f_i(a)$. Is h continuous? Give a proof or counterexample.