Journal of Knot Theory and Its Ramifications, Vol. 10, No. 8 (2001) 1143–1154

INTRISICALLY n-LINKED GRAPHS

ERICA FLAPAN* and JAMES POMMERSHEIM†
Department of Mathematics, Pomona College, Claremont, CA 91711, USA
* *eflapan@pomona.edu*
† *jpommersheim@pomona.edu*

JOEL FOISY
Department of Mathematics, SUNY Potsdam, Potsdam, NY 13676, USA
foisyjs@potsdam.edu

RAMIN NAIMI
Department of Mathematics, Occidental College, Los Angeles, CA 90041, USA
rnaimi@oxy.edu

Received 6 May 2000
Revised 12 April 2001

ABSTRACT

For every natural number n, we exhibit a graph with the property that every embedding of it in $\mathbb{R}^3$ contains a non-split n-component link. Furthermore, we prove that our graph is minor minimal in the sense that every minor of it has an embedding in $\mathbb{R}^3$ that contains no non-split n-component link.

1. Introduction

Conway and Gordon [CG] and Sachs [Sa] proved that the complete graph on six vertices, K_6, is *intrinsically linked*; that is, every embedding of K_6 in $\mathbb{R}^3$ contains a homologically non-trivial link of two components. Furthermore, Sachs showed that every graph which can be obtained from K_6 by a finite sequence of ΔY moves (i.e., replacing a triangle by a Y) and a finite sequence of $Y\Delta$ moves (i.e., replacing a Y by a triangle) is also intrinsically linked. The graph K_6 together with the six graphs that can be obtained in this way are known as the *Petersen family* of graphs. A graph H is said to be a *minor* of a graph G if H can be obtained from G by deleting and/or contracting a finite number of edges. A graph G is said to be *minor minimal* with respect to a property if G has the property, but no minor of G has the property. Robertson, Seymour, and Thomas [RST] proved that a graph is minor minimal with respect to being intrinsically linked if and only if it is in the Petersen family.

The concept of an intrinsically linked graph has a natural generalization to links of n components. We say that a link L is *split* if there is an embedding of a 2-sphere F in $\mathbb{R}^3 - L$ such that each component of $\mathbb{R}^3 - F$ contains at least one component of L.

Definition. For any natural number n, we define a graph G to be *intrinsically n-linked* if every embedding of G in $\mathbb{R}^3$ contains a non-split link of n components.

As stated above, the results of [RST] completely characterize intrinsically 2-linked graphs. Intrinsically 3-linked graphs were investigated in [FNP], where it was shown that K_{10} is the smallest complete graph that is intrinsically 3-linked. However, the question of whether K_{10} is minor minimal with respect to being intrinsically 3-linked was left open,

1991 *Mathematics Subject Classification.* 57M25, 57M15.
Key words and phrases. intrinsically linked, self-linked, embedded graphs, spatial graphs, spatial embeddings, minor minimal.

and indeed no minor minimal intrinsically 3-linked graph was given. In this paper, for each $n \geqslant 2$, we construct a graph that is intrinsically n-linked, and which is minor minimal with respect to this property. In particular we prove the following theorems.

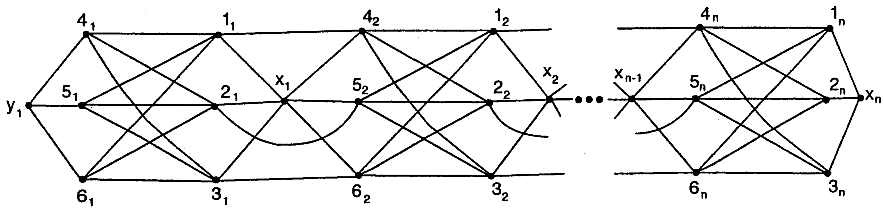

FIGURE 1. The graph $G(n)$ is intrinsically $(n+1)$-linked

Theorem 1. *For every natural number n, let $G(n)$ denote the graph illustrated in Figure 1. Then $G(n)$ is intrinsically $(n+1)$-linked.*

Theorem 2. *For every natural number n, the graph $G(n)$ is minor minimal with respect to being intrinsically $(n+1)$-linked.*

One particular type of non-split link resembles the chain of a necklace. Formally, we define an *n-necklace* to be a link $L_1 \cup L_2 \cup \cdots \cup L_n$ such that for each $i = 1, \ldots, n-1$, $L_i \cup L_{i+1}$ is non-split and $L_n \cup L_1$ is non-split. For each $n \geqslant 2$, we construct a graph $F(n)$ which has the property that every embedding of $F(n)$ in $\mathbb{R}^3$ contains an n-necklace. In particular the following is a corollary to the proof of Theorem 1.

Corollary. *For every $n \geqslant 3$, let $F(n)$ denote the graph illustrated in Figure 2. Then every embedding of $F(n)$ in $\mathbb{R}^3$ contains an n-necklace.*

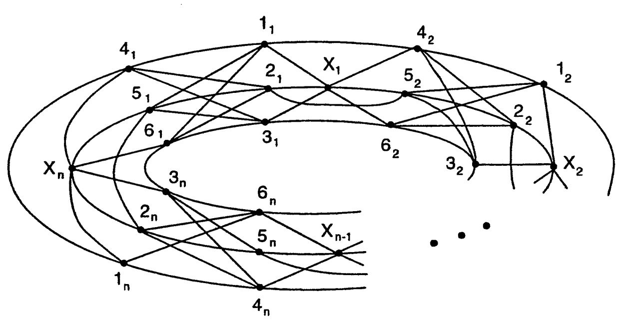

FIGURE 2. Every embedding of the graph $F(n)$ contains an n-necklace

By definition, a 3-necklace $L_1 \cup L_2 \cup L_3$ is pairwise non-split in the sense that for each $i \neq j$, $L_i \cup L_j$ is non-split. The graph $F(3)$ is the first known example of a graph such that every embedding of it contains a pairwise non-split link of 3-components. The existence of such a graph suggests the problem of finding a graph $Q(n)$ for each $n > 3$ such that every embedding of $Q(n)$ contains a pairwise non-split link of n-components.

Motwani, Raghunathan, and Saran [MRS] proved that if a graph G is intrinsically linked, and a graph G' is obtained from G by replacing a triangle in G by a Y, then G'

is also intrinsically linked. It is easy to see that their proof can be modified to show that the analogous statement is true for intrinsically n-linked graphs. Thus all of the graphs that can be obtained from $G(n)$ by replacing finitely many triangles by Y's will also be intrinsically $(n+1)$-linked.

Since $G(1)$ is one of the Petersen graphs (cf. [Sa]), it follows from results of [RST] (stated above) that every minor minimal intrinsically 2-linked graph can be obtained from $G(1)$ by a finite sequence of ΔY and $Y\Delta$ moves. One might ask if the analogous statement holds for intrinsically n-linked graphs. That is, can every minor minimal intrinsically n-linked graph be obtained from $G(n-1)$ by a finite sequence of ΔY and $Y\Delta$ moves? This is not the case in general. In the final section of this paper we exhibit a minor minimal intrinsically 3-linked graph that cannot be obtained from $G(2)$ by a finite sequence of ΔY and $Y\Delta$ moves. Nonetheless, it follows from [RS] that for every $n \in \mathbb{N}$, there are only finitely many minor minimal intrinsically n-linked graphs. For $n \geqslant 3$, it would be interesting to determine the complete list of such graphs.

2. Construction of the graphs $G(n)$

Let E denote the graph that is illustrated in Figure 2. The reader will notice that this graph is the same as $G(1)$, which was introduced above. (Sachs [Sa] refers to this graph as $K^-_{4,4}$). This graph is one of the Petersen graphs and hence is minor minimal with respect to being intrinsically 2-linked. Figure 3 illustrates a particular embedding of $G(1)$ in $\mathbb{R}^3$, which will be useful in the proof of Theorem 2.

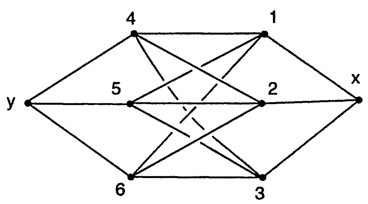

FIGURE 3. The graph E is intrinsically linked

The graph $G(n)$ can be constructed as follows. We start with n distinct copies of E, which we will denote by $E_1, \dots, E_n$, where the vertices of E_i are labelled by $1_i, \dots, 6_i$, x_i and y_i corresponding to the labeling of the vertices of E. We create the graph $G(n)$ from $\bigcup_{i=1}^n E_i$ by identifying the pair of vertices x_i and y_{i+1}, and adding an edge between the pair of vertices 1_i and 4_{i+1}, between the pair of vertices 2_i and 5_{i+1}, and between the pair of vertices 3_i and 6_{i+1}, for each $i = 1, \dots, n-1$. We illustrate an embedding of $G(2)$ in Figure 4. A non-split link of three components is highlighted.

Observe that in Figure 4, if we ignore the vertex labels then the embedding of E_2 can be obtained from the embedding of E_1 by performing a rotation by 180°. For any natural number n, we create an embedding of $G(n)$ as follows. Start with the embedding of E_1 that is illustrated in Figure 3. Then for each $i = 2, \dots, n$, sequentially embed E_i by rotating the embedding of E_{i-1} by 180°. In Figure 5, we illustrate this embedding of $G(3)$ with a non-split link of four components highlighted.

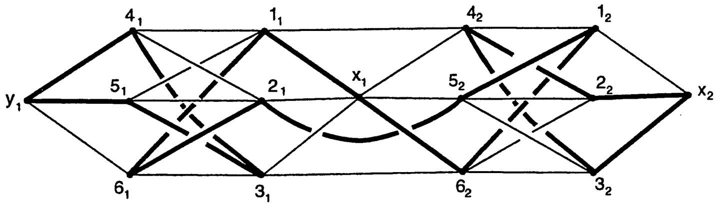

FIGURE 4. An embedding of $G(2)$ with a non-split link of three components highlighted

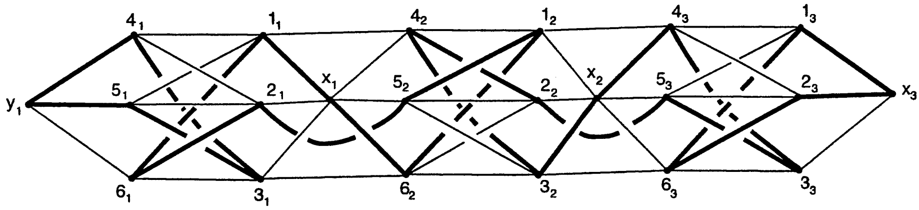

FIGURE 5. An embedding of $G(3)$ with a non-split link of four components highlighted

By inspection we can see that for any n, this embedding of $G(n)$ contains a non-split link of $(n+1)$ components. In Section 3, we will prove that every embedding of $G(n)$ contains a non-split link of $(n+1)$ components.

3. $G(n)$ IS INTRINSICALLY $(n+1)$-LINKED

To prove Theorem 1, we will actually prove a slightly stronger statement. We will show that every embedding of $G(n)$ contains a link $L = L_1 \cup L_2 \cup \cdots \cup L_{n+1}$ such that each L_i has non-zero mod 2 linking number with L_{i+1}. For any pair of disjoint simple closed curves A and B in $\mathbb{R}^3$, we denote the mod 2 linking number of A and B by $\omega(A, B)$. With this notation in hand, we prove the following lemma using an elementary homology argument.

Lemma 1. *Let G be a graph embedded in $\mathbb{R}^3$ containing simple closed curves C_1, C_2, C_3, and C_4 such that the following properties hold.*

a) C_1 and C_4 are disjoint from each other and from both C_2 and C_3.

b) C_2 and C_3 intersect in precisely one vertex x.

c) $\omega(C_1, C_2) = 1$ and $\omega(C_3, C_4) = 1$.

d) There are vertices $u \neq x$ in C_2 and $v \neq x$ in C_3 and a path P in G with endpoints u and v whose interior is disjoint from each of the C_i.

Then there exists a simple closed curve $S \subset (C_2 \cup C_3 \cup P)$ such that $\omega(S, C_1) = 1$ and $\omega(S, C_4) = 1$.

Proof. We know that $[C_2]$ is non-trivial in $H_1(\mathbb{R}^3 - C_1; \mathbb{Z}_2)$, and $[C_3]$ is non-trivial in $H_1(\mathbb{R}^3 - C_4; \mathbb{Z}_2)$. If either $[C_2]$ is non-trivial in $H_1(\mathbb{R}^3 - C_4; \mathbb{Z}_2)$, or $[C_3]$ is non-trivial in $H_1(\mathbb{R}^3 - C_1; \mathbb{Z}_2)$, then we are done. So we assume that this is not the case. Let D_2 denote a path in C_2 with endpoints u and x, and let D_3 denote a path in C_3 with endpoints v and x. Let $A = D_2 \cup D_3 \cup P$.

First suppose that $[A]$ is non-trivial in either $H_1(\mathbb{R}^3 - C_1; \mathbb{Z}_2)$ or $H_1(\mathbb{R}^3 - C_4; \mathbb{Z}_2)$. Without loss of generality we assume that $[A]$ is non-trivial in $H_1(\mathbb{R}^3 - C_1; \mathbb{Z}_2)$. If $[A]$ is also non-trivial in $H_1(\mathbb{R}^3 - C_4; \mathbb{Z}_2)$, then we can let $S = A$. So we assume that $[A]$ is trivial in $H_1(\mathbb{R}^3 - C_4; \mathbb{Z}_2)$. Let B_3 denote the simple closed curve obtained from $C_3 \cup A$ by omitting the interior of the arc D_3. In $H_1(\mathbb{R}^3 - C_4; \mathbb{Z}_2)$ we have the equation $[C_3] + [A] = [B_3]$. Thus $[B_3]$ must be non-trivial in $H_1(\mathbb{R}^3 - C_4; \mathbb{Z}_2)$. In $H_1(\mathbb{R}^3 - C_1; \mathbb{Z}_2)$ we also have the equation $[C_3] + [A] = [B_3]$. Thus $[B_3]$ must also be non-trivial in $H_1(\mathbb{R}^3 - C_4; \mathbb{Z}_2)$. So we can let $S = B_3$.

So now we assume that $[A]$ is trivial in both $H_1(\mathbb{R}^3 - C_1; \mathbb{Z}_2)$ and $H_1(\mathbb{R}^3 - C_4; \mathbb{Z}_2)$. Let S denote the simple closed curve obtained from $C_2 \cup C_3 \cup P$ by removing the interiors of the arcs D_2 and D_3. Then we have the equation $[C_2] + [C_3] + [A] + [S] = 0$ in both $H_1(\mathbb{R}^3 - C_1; \mathbb{Z}_2)$ and $H_1(\mathbb{R}^3 - C_4; \mathbb{Z}_2)$. It follows that $[S]$ is non-trivial in both $H_1(\mathbb{R}^3 - C_1; \mathbb{Z}_2)$ and $H_1(\mathbb{R}^3 - C_4; \mathbb{Z}_2)$. □

We are now ready to prove Theorem 1, which states that the graph $G(n)$ is intrinsically $(n+1)$-linked.

Proof of Theorem 1. We will use induction on n to prove the following more specific statement, from which our theorem follows.

Claim: Every embedding of $G(n)$ in $\mathbb{R}^3$ contains an $(n+1)$-component link $L = L_1 \cup L_2 \cup \cdots \cup L_{n+1}$ such that L_1 contains vertex y_1, L_{n+1} contains vertex x_n, and $\omega(L_i, L_{i+1}) = 1$ for $i = 1, \ldots, n$.

Proof of Claim: For $n = 1$, we know from Sachs' Theorem [Sa] that every embedding of $G(1)$ in $\mathbb{R}^3$ necessarily contains a link $L = L_1 \cup L_2$ such that $\omega(L_1, L_2) = 1$ and L_1 contains y_1 and L_2 contains x_1.

We now assume the claim is true for $G(n-1)$ and show that it holds for $G(n)$. Suppose that $G(n)$ is embedded in $\mathbb{R}^3$. Then $G(n-1) \subset G(n)$ contains a link $L = L_1 \cup L_2 \cup \cdots \cup L_n$ such that L_1 contains the vertex y_1, L_n contains the vertex x_{n-1} and $\omega(L_i, L_{i+1}) = 1$ for $i = 1, \ldots, n-1$. Also, E_n contains a 2-component link, whose components we denote by C_3 and C_4, such that $\omega(C_3, C_4) = 1$ and C_3 contains x_{n-1} and C_4 contains x_n. Now since L_n is contained in $G(n-1)$ and contains x_{n-1}, it must contain at least two of the vertices 1_{n-1}, 2_{n-1}, or 3_{n-1}. Also since C_3 is contained in E_n and contains x_{n-1} it must contain at least two of the vertices 4_n, 5_n, or 6_n. Thus $L_n \cup C_3$ must contain one of the pairs of vertices $\{1_{n-1}, 4_n\}$, $\{2_{n-1}, 5_n\}$, or $\{3_{n-1}, 6_n\}$. Without loss of generality, $L_n \cup C_3$ contains $\{1_{n-1}, 4_n\}$.

Now let $C_1 = L_{n-1}$, $C_2 = L_n$, $u = 1_{n-1}$, $v = 4_n$, and $P = \overline{1_{n-1}4_n}$. We want to apply Lemma 1 to C_1, C_2, C_3, and C_4, but first we make some observations. Since C_1 is contained in $G(n-1) - \{x_{n-1}\}$ and C_3 and C_4 are contained in E_n, C_1 is disjoint from C_3 and C_4. Also because C_1 and C_2 are components of L they are disjoint, and since C_3 and C_4 are components of a link they are disjoint. Finally since C_4 is contained in $E_n - \{x_{n-1}\}$ and C_2 is contained in $G(n-1)$, C_4 is disjoint from C_2. Furthermore, since C_2 is contained in $G(n-1)$ and C_3 is contained in E_n they intersect only at the vertex x_{n-1}. Thus we can apply Lemma 1 to conclude that there is a simple closed curve S in $C_2 \cup C_3 \cup P$ such that $\omega(L_{n-1}, S) = \omega(C_1, S) = 1$ and $\omega(C_4, S) = 1$. Let $K = L_1 \cup L_2 \cup \cdots \cup L_{n-1} \cup S \cup C_4$. Then K is an $(n+1)$-component link that satisfies our claim.

Hence we have proved our claim, and thus the theorem is proven as well. □

4. Every embedding of $F(n)$ contains an n-necklace

We now construct the graph $F(n)$ as follows. We start with $G(n)$ and identify the vertices y_1 and x_n, to a vertex which we still call x_n. Then we add edges between the pair of vertices $\{1_n, 4_1\}$, between the pair of vertices $\{2_n, 5_1\}$, and between the pair of vertices $\{3_n, 6_1\}$. The graph $F(n)$ is illustrated in Figure 2. The corollary below follows from the proof of Theorem 1.

Corollary. *For $n \geqslant 3$, let $F(n)$ denote the graph illustrated in Figure 2. Then every embedding of $F(n)$ in $\mathbb{R}^3$ contains an n-necklace.*

Proof. Let $F(n)$ be embedded in $\mathbb{R}^3$. We choose an expansion of the vertex x_n to an edge e with vertices y_1 and x_n such that y_1 is connected to vertices $\{4_1, 5_1, 6_1\}$ and x_n is connected to vertices $\{1_n, 2_n, 3_n\}$. This embedded expansion determines an embedding of the subgraph $G(n)$. By the proof of Theorem 1, $G(n)$ contains an $(n+1)$-component link $L_1 \cup L_2 \cup \cdots \cup L_{n+1}$ such that L_1 contains the vertex y_1, L_{n+1} contains the vertex x_n and $\omega(L_i, L_{i+1}) = 1$ for $i = 1, \ldots, n$. From here the proof is similar to that of Theorem 1. Since L_1 contains y_1 it must contain two of the vertices $\{4_1, 5_1, 6_1\}$, and since L_{n+1} contains x_n it must contain two of the vertices $\{1_n, 2_n, 3_n\}$. Thus without loss of generality, $L_1 \cup L_{n+1}$ contains the pair $\{5_1, 2_n\}$. By hypothesis $n \geqslant 3$, hence L_2 is disjoint from L_n.

We now collapse the edge e, so that we have our original embedding of the graph $F(n)$. In this embedding L_1 and L_n share the vertex x_n. Now we apply Lemma 1 where $C_1 = L_2$, $C_2 = L_1$, $C_3 = L_{n+1}$, $C_4 = L_n$, $u = 5_1$, $v = 2_n$, and $p = \overline{5_1 2_n}$. Thus we get a simple closed curve K_1 in $L_1 \cup L_{n+1} \cup p$ such that $\omega(K_1, L_2) = \omega(K_1, L_n) = 1$. Now $K_1 \cup L_2 \cup \cdots \cup L_n$ is an n-necklace. □

5. No minor of $G(n)$ is intrinsically $(n+1)$-linked

We shall begin by considering the particular embedding of $G(n)$ that we described in Section 2. So as to avoid confusing the abstract graph $G(n)$ with this particular embedding of it in $\mathbb{R}^3$, we shall refer to the embedded graph as $H(n)$, and for each i, the particular embedding of E_i will be denoted by F_i. In Figure 6, we illustrate $H(n)$ where n is even and a particular non-split link of $(n+1)$ components has been highlighted.

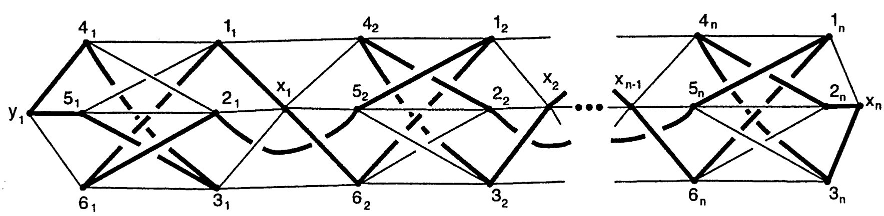

FIGURE 6. For n even, $H(n)$ contains this non-split link of $n+1$ components

In Figure 7, we illustrate $H(n)$ where n is odd and the analogous non-split link of $n+1$ components has been highlighted.

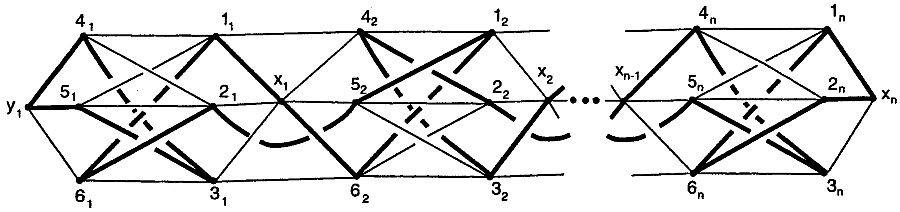

FIGURE 7. For n odd, $H(n)$ contains this non-split link of $n+1$ components

For any n, we shall denote this link by $L(n+1)$ and the components of $L(n+1)$ by $L_1, \dots, L_{n+1}$, from left to right as they appear in Figures 6 and 7. Observe that L_1 is contained in F_1, L_{n+1} is contained in F_n, and for each $i = 2, \dots, n$, the component L_i is contained in $F_{i-1} \cup F_i$.

We will first prove a lemma which shows that $L_1 \cup \cdots \cup L_{n+1}$ is the only $(n+1)$-component non-split link in $H(n)$. Then we will use this lemma to prove Theorem 2 by showing that for any edge e in $G(n)$ there is an automorphism of $G(n)$ which takes e to an edge f such that $H(n) - f$ (i.e. $H(n)$ with the edge f removed) will contain no $(n+1)$-component link. Similarly, for any edge e in $G(n)$, we will show that there is an automorphism of $G(n)$ which takes e to an edge f such that $H(n)/f$ (i.e. $H(n)$ with the edge f contracted) will contain no $(n+1)$-component link.

Lemma 2. *For each natural number n, $L(n+1)$ is the only $(n+1)$-component non-split link in $H(n)$, and $H(n)$ contains no non-split link of more than $n+1$ components.*

Proof. We use induction on n. Figure 8 illustrates $H(1)$ with the link $L(2) = L_1 \cup L_2$ highlighted. Observe that each component of any link in $H(1)$ must contain four vertices since there are no triangles in $G(1)$. Thus any link in $H(1)$ must contain all eight vertices. By inspection we can see that the highlighted simple closed curve $y_1 5_1 3_1 4_1$ is the only 4-edge cycle containing y_1 that does not bound a disk in $\mathbb{R}^3 - H(1)$. It follows that $L(2)$ is the unique non-split link of two components in $H(1)$. Furthermore, it is clear that $H(1)$ contains no link of more than two components.

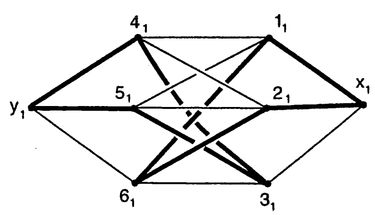

FIGURE 8. $L(2)$ is the unique non-split link of two components in $H(1)$

Now assume that $L(n)$ is the unique n-component non-split link contained in $H(n-1)$, and $H(n-1)$ contains no non-split link of more than n components. Suppose that $H(n)$ contains some non-split link $K = K_1 \cup \cdots \cup K_m$ where $m \geqslant n+1$. It suffices to show that $K = L(n+1)$.

From here our proof will proceed according to the following outline. First we will show that K is isotopic to some non-split link $J = J_1 \cup \cdots \cup J_m$ no component of which contains two edges of any triangle in $H(n)$. Then we will prove that there is some component of J that is not contained entirely in either $H(n-1)$ or F_n. Next we will use the inductive hypothesis to prove that there is a component which is entirely contained in F_n. Then we will prove that if we remove the component of J which is contained in F_n, we will obtain a non-split link of n components. This, together with our inductive hypothesis, will enable us to show that $J = L(n+1)$. Finally, we will prove that, in fact, $K = J$.

We begin by observing that in $H(n)$, for every $j = 2, \ldots, n$, the triangles $1_{j-1}x_{j-1}4_j$, $2_{j-1}x_{j-1}5_j$, and $3_{j-1}x_{j-1}6_j$ each bound a disk whose interior is in $\mathbb{R}^3 - H(n)$. Thus none of these triangles is a component of our non-split link K. Suppose that some component K_i of K contains two of the edges of one of these triangles. Let K_i' denote the simple closed curve that is obtained from K_i by replacing these two edges of a triangle by the third side of the triangle. Note that K_i' is ambient isotopic to K_i in $\mathbb{R}^3 - (K - K_i)$. Let J denote the link we obtain from K by replacing each K_i that contains two sides of a triangle by K_i'. (If there are no components K_i that contain two sides of a triangle, we let $J = K$.) Let the components of J be denoted by $J_1, \ldots, J_m$. Then J is an m-component non-split link no component of which contains two sides of a single triangle. It follows that if any component J_i contains precisely one of the vertices 1_j, 2_j, 3_j, then J_i is contained in $H(j)$. Similarly if any component J_i contains precisely one of the vertices 4_j, 5_j, or 6_j, then J_i is disjoint from $H(j-1)$.

Now, since $H(n-1)$ contains no non-split link of more than n components, J cannot be contained in $H(n-1)$. Thus J has some component that is not contained in $H(n-1)$. Suppose that every component J_i that is not contained in $H(n-1)$ is contained in F_n. Then J would be split. Hence there is some component, say J_n that is not contained entirely in either $H(n-1)$ or in F_n. Thus J_n contains at least two of the vertices 1_{n-1}, 2_{n-1}, and 3_{n-1}, and at least two of the vertices 4_n, 5_n, and 6_n.

Since at most one of the vertices 4_n, 5_n, and 6_n is not contained in J_n and no component of J contains two sides of a triangle, every component of $J - J_n$ must be contained in either $H(n-1)$ or in F_n. Also, since at most one of the vertices 1_{n-1}, 2_{n-1}, and 3_{n-1} is not contained in J_n, no component of $J - J_n$ can contain the vertex x_{n-1} because no component contains two sides of a triangle. Suppose that no component of J is entirely contained in F_n. Then $J - J_n$ is contained in $H(n-1) - \{x_{n-1}\}$. Therefore, J_n is ambient isotopic in $\mathbb{R}^3 - (J - J_n)$ to a simple closed curve in $H(n-1)$. This implies that there is a non-split link of $m \geqslant n+1$ components in $H(n-1)$. As this contradicts our inductive hypothesis, we can conclude that there must be a component of J, say J_{n+1}, that is entirely contained in F_n.

Now, as J_n contains at least two of the vertices 4_n, 5_n, and 6_n, it must also contain at least one of the vertices 1_n, 2_n, and 3_n. There is no simple closed curve in F_n that contains only three vertices. Since J_{n+1} is contained in F_n and cannot contain x_{n-1}, it must contain one vertex from among 4_n, 5_n, and 6_n, two vertices from among 1_n, 2_n, and

3_n, and the vertex x_n. By inspection we can see that if n is even, then the only 4-edge cycle in F_n containing the vertex x_n which does not bound a disk in $\mathbb{R}^3 - H(n)$ is $x_n 3_n 4_n 2_n$ (highlighted in Figure 6). If n is odd, then the only 4-edge cycle in F_n containing the vertex x_n that does not bound a disk in $\mathbb{R}^3 - H(n)$ is $x_n 1_n 6_n 2_n$ (highlighted in Figure 7). Thus $J_{n+1} = x_n 3_n 4_n 2_n$ when n is even and $J_{n+1} = x_n 1_n 6_n 2_n$ when n is odd. Furthermore, when n is even, J_n must contain the vertices 5_n, 6_n, and 1_n; and when n is odd, J_n must contain the vertices 5_n, 4_n, and 3_n.

We will consider the links obtained from J by removing the components J_{n+1} and J_n. Let $J' = J - J_{n+1}$ and let $J'' = J' - J_n$.

Claim: J' is a non-split link of n components, and $m = n + 1$.

Proof of Claim: Suppose that the link J' is split. We will show that this assumption implies that J is split. Since J' is split, there is a 2-sphere F in $\mathbb{R}^3 - J'$ with some component of J' in each component of $\mathbb{R}^3 - F$. Let the components of $\mathbb{R}^3 - F$ be denoted by A and B. Then A and B each contain at least one component of J'. Now without loss of generality J_n is contained in A and for some $q \notin \{n, n+1\}$, the component J_q is contained in B. Observe that J_{n+1} bounds a disk D which is disjoint from $J_1 \cup \cdots \cup J_{n-1}$ and is punctured once by J_n. If J_{n+1} were disjoint from F then J would be split. So we can assume that J_{n+1} intersects F. However, using the disk D and a standard innermost disk and arc argument, we can isotop F to a 2-sphere F' which is disjoint from J_{n+1}. Furthermore, F' is disjoint from J, and one component of $\mathbb{R}^3 - F'$ contains $J_n \cup J_{n+1}$ while the other component of $\mathbb{R}^3 - F'$ contains J_q. It follows that J is a split link. As this contradicts our hypothesis we conclude that J' is a non-split link of $m - 1$ components.

Recall that every component of J'' is contained in $H(n-1)$, and J_n is ambient isotopic in $\mathbb{R}^3 - J''$ to a simple closed curve P in $H(n-1)$. Thus $J'' \cup P$ is a non-split link of $m-1$ components which is entirely contained in $H(n-1)$. Now by our inductive hypothesis, we must have $m - 1 \leqslant n$. By hypothesis $m \geqslant n+1$, hence in fact $m = n+1$ and our claim is proven.

Recall that at most one of the vertices 1_{n-1}, 2_{n-1}, and 3_{n-1} is not contained in J_n, so no component of J'' can contain the vertex x_{n-1}. Now, J_n is ambient isotopic in $\mathbb{R}^3 - J''$ to the simple closed curve P in $H(n-1)$ that contains the same vertices of $H(n-1)$ as J_n does. So $J'' \cup P$ is a non-split link of n components which is contained in $H(n-1)$. Thus by our inductive hypothesis $J'' \cup P = L(n)$. If $n-1$ is odd, then $P = x_{n-1}1_{n-1}6_{n-1}2_{n-1}$; and if $n-1$ is even, then $P = x_{n-1}3_{n-1}4_{n-1}2_{n-1}$. Thus if $n-1$ is odd, $J_n = x_{n-1}1_{n-1}6_{n-1}2_{n-1}5_n 1_n 6_n$. On the other hand, if $n-1$ is even, $J_n = x_{n-1}3_{n-1}4_{n-1}2_{n-1}5_n 3_n 4_n$. It follows that $J = L(n+1)$.

Suppose that $K \neq J$; then some component K_i of K contains two sides of a triangle, while the component J_i which replaces K_i in J contains only one side of that triangle. In particular, this means that J contains fewer vertices than K. As $J = L(n+1)$ contains every vertex of $H(n)$, we must have $K = J = L(n+1)$. □

Now we shall use Lemma 2 to prove Theorem 2.

Theorem 2. *For every natural number n, $G(n)$ is minor minimal with respect to being intrinsically $(n+1)$-linked.*

Proof. Theorem 1 states that $G(n)$ is intrinsically $(n+1)$-linked. Now we will show that no minor of $G(n)$ is intrinsically $(n+1)$-linked by showing that for every edge e in $G(n)$ there are embeddings of $G(n)-e$ and $G(n)/e$ that contain no non-split link of $n+1$ components.

Up to automorphism of $G(n)$, the edges of $G(n)$ can be grouped into the following types. Edges of type a are those with one endpoint which is either y_1 or x_n. Edges of type b_i are those with one endpoint in the set $\{1_i, 2_i, 3_i\}$, and the other endpoint in the set $\{4_i, 5_i, 6_i\}$. Edges of type c_i are those with one endpoint which is x_i and $i \neq n$. Edges of type d_i are those of the form $1_i 4_{i+1}$, $2_i 5_{i+1}$, and $3_i 6_{i+1}$.

We will start with the embedding $H(n)$, and show that for each of the above types of edges we can remove some edge e of that type such that $H(n)-e$ will contain no non-split link of $n+1$ components. Then we will show that for each of the above types of edges, we can collapse some edge e of that type such that $H(n)/e$ will contain no non-split link of $n+1$ components. Since for every two edges of the same type there is an automorphism of $G(n)$ taking one to the other, it is enough to pick one edge from each type to prove the theorem.

For each type, we choose to remove the following edge e. For type a, we choose $e = \overline{y_1 4_1}$. For type b_i, we choose $e = \overline{4_i 3_i}$. For type c_i, we choose $e = \overline{x_i 1_i}$ if i is odd and $e = \overline{x_i 3_i}$ if i is even. For type d_i, we choose $e = \overline{2_i 5_{i+1}}$. Each of these edges is contained in $L(n+1)$. By Lemma 2, $L(n+1)$ is the only non-split link of $n+1$ components in $H(n)$. Hence if we remove any of the above edges e from $H(n)$, the graph will contain no non-split $(n+1)$-component link.

Now we will consider $H(n)/e$ for each type of edge e. First, note that if e is in a triangle, then collapsing e will create two edges with the same vertices. In this case the definition of minor requires that one of these two edges be omitted. We have seen that every triangle in $H(n)$ bounds a disk whose interior is in the complement of $H(n)$. Thus, if the edge e is contained in a triangle in $H(n)$, then collapsing e gives a pair of edges which cobound a disk. Hence it makes no difference which of these two edges we omit from the embedding of $H(n)/e$. Thus, the embedding of $H(n)/e$ is well defined up to isotopy.

For each type of edge, we choose to collapse the following edge e. For type a, we choose $e = \overline{y_1 6_1}$. For type b_i, we choose $e = \overline{1_i 4_i}$. For type c_i, we choose $e = \overline{x_i 3_i}$ if i is odd and $e = \overline{x_i 1_i}$ if i is even. Finally, for type d_i, we choose $e = \overline{1_i 4_{i+1}}$. Observe that each of these edges e has one endpoint v in one component of $L(n+1)$ and the other endpoint w in a different component of $L(n+1)$. Suppose that $H(n)/e$ contains a non-split link Q of $n+1$ components. If no component of Q contains the collapsed vertex vw then Q is contained in $H(n) - \{v, w\}$. As $L(n+1)$ contains every vertex of $H(n)$ this is not possible. So some component Q_i of Q must contain the collapsed vertex vw. Then $H(n)$ will contain a non-split link R of $n+1$ components, where R is identical to Q except that the component Q_i has been replaced by a simple closed curve R_i in $H(n)$ such that R_i contains either the vertex v, the vertex w, or the vertices v and w and the edge $e = \overline{vw}$. In any case, no component of $R - R_i$ will contain the vertex v or the vertex w. This is impossible, since $L(n+1)$ is the only non-split link of $n+1$ components in $H(n)$, and v and w are contained in different components of $L(n+1)$. Thus $H(n)/e$ contains no non-split link of $n+1$ components.

Hence no minor of $G(n)$ is intrinsically $(n+1)$-linked, and so $G(n)$ is minor minimal with respect to being intrinsically $(n+1)$-linked. □

6. Minor minimal intrinsically 3-linked graphs

Robertson, Seymour, and Thomas [RST] proved that any intrinsically 2-linked graph contains one of the Petersen graphs as a minor. Each of the Petersen graphs can be obtained from any one of the Petersen graphs (and from $G(1)$ in particular) by a finite a sequence of ΔY and $Y\Delta$ moves. It is natural to wonder whether the complete list of graphs that are minor minimal with respect to being intrinsically 3-linked can be obtained from our graph $G(2)$ by a finite sequence of such moves. We will now show as follows that this is not possible.

Consider the graph J that is illustrated in Figure 9. We will first show that J is intrinsically 3-linked, and then show that J cannot have a minor which is obtained from $G(2)$ by a finite sequence of ΔY and $Y\Delta$ moves.

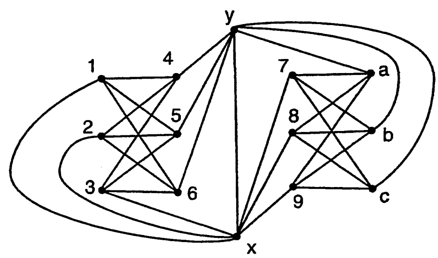

FIGURE 9. The graph J is intrinsically 3-linked

Suppose that J is embedded in $\mathbb{R}^3$. Let J_1 denote the subgraph of J with vertices 1, 2, 3, 4, 5, 6, x, and y. Let J_2 denote the subgraph of J with vertices 7, 8, 9, a, b, c, x, and y. Let $e = \overline{xy}$. We now collapse the edge e in the embedded graphs J_1 and J_2. Observe that J_1/e is isomorphic to the graph $K_{3,3,1}$, and hence by Sachs' Theorem [Sa] contains a square S_1 and a triangle T_1 such that $\omega(S_1, T_1) = 1$. Let z denote the vertex obtained from x and y by collapsing the edge e. Then z is necessarily contained in T_1. Observe that in J_1 x is adjacent to 1, 2, and 3, but not to 4, 5, or 6, and y is adjacent to 4, 5, and 6 but not to 1, 2, or 3. The triangle T_1 in J_1/e comes from a square K_1 in J_1 by collapsing the edge e, and the square S_1 is contained in J_1 and is disjoint from K_1. It follows that $e \subset K_1$ and $\omega(S_1, K_1) = 1$. By a similar argument in J_2/e we obtain a pair of disjoint squares S_2 and K_2 in J_2 such that $e \subset K_2$ and $\omega(S_2, K_2) = 1$. Observe that $K_1 \cap K_2 = e$, $S_1 \cap K_2 = \phi$, $S_2 \cap K_1 = \phi$, and $S_1 \cap S_2 = \phi$.

Now we apply the following elementary lemma from [FNP], whose proof is similar to the proof of our Lemma 1. It follows from this lemma that J has a non-split link of three components. Hence J is intrinsically 3-linked.

Lemma [FNP]. *Suppose J is a graph that is embedded in $\mathbb{R}^3$, and contains simple closed curves S_1, K_1, K_2, and S_2. Suppose that S_1 and S_2 are disjoint from each other and both are disjoint from K_1 and K_2, and $K_1 \cap K_2$ is an arc. If $\omega(S_1, K_1) = 1$ and $\omega(S_2, K_2) = 1$, then J contains a non-split link of three components.*

It now follows that J contains some minor M which is minor minimal with respect to being intrinsically 3-linked. Observe that J has 31 edges, hence M has no more than 31 edges. Performing $Y\Delta$ and ΔY moves on a graph does not change the total number of edges in the graph. Thus, since $G(2)$ has 33 edges, M cannot be obtained from $G(2)$ by a finite sequence of ΔY and $Y\Delta$ moves.

It follows from this example that the complete list of graphs which are minor minimal with respect to being intrinsically 3-linked cannot be obtained from $G(2)$ by a finite sequence of such moves. This suggests that finding this list may be a difficult problem.

REFERENCES

[CG] J. Conway, C. McA Gordon, *Knots and links in spatial graphs*, J. of Graph Theory **7** (1983), 445-453.

[FNP] E. Flapan, R. Naimi, J. Pommersheim, *Intrinsically triple linked complete graphs*, to appear in Topology and its Applications.

[MRS] R. Motwani, A. Raghunathan, H. Saran, *Constructive results from graph minors: Linkless embeddings*, 29th Annual Symposium on Foundations of Computer Science, IEEE, 1988, pp. 398-409.

[Sa] H. Sachs, *On spatial representations of finite graphs*, Colloq. Math. Soc. János Bolyai (A. Hajnal, L. Lovasz, V.T. Sós, eds.), vol. 37, North Holland, Amsterdam, New York, 1984, pp. 649-662.

[RS] N. Robertson, P. Seymour, *Graph minors XVI. Wagner's conjecture*, preprint.

[RST] N. Robertson, P. Seymour, R. Thomas, *Sachs' linkless embedding conjecture*, Journal of Combinatorial Theory, Series B **64** (1995), 185-227.

ERICA FLAPAN, DEPARTMENT OF MATHEMATICS, POMONA COLLEGE, CLAREMONT, CA 91711, U.S.A.
E-mail address: eflapan@pomona.edu

JOEL FOISY, DEPARTMENT OF MATHEMATICS, SUNY POTSDAM, POTSDAM, NY, 13676, U.S.A.
E-mail address: foisyjs@potsdam.edu

RAMIN NAIMI, DEPARTMENT OF MATHEMATICS, OCCIDENTAL COLLEGE, LOS ANGELES, CA 90041, U.S.A.
E-mail address: rnaimi@oxy.edu

JAMES POMMERSHEIM, DEPARTMENT OF MATHEMATICS, POMONA COLLEGE, CLAREMONT, CA 91711, U.S.A.
E-mail address: jpommersheim@pomona.edu