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THE Y-TRIANGLE MOVE DOES NOT PRESERVE INTRINSIC KNOTTEDNESS

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Abstract

We answer the question "Does the Y-triangle move preserve intrinsic knottedness?" in the negative by giving an example of a graph that is obtained from the intrinsically knotted graph K_7 by triangle-Y and Y-triangle moves but is not intrinsically knotted.

1. Introduction

A graph is said to be *intrinsically knotted* (IK) if every embedding of it in $\mathbb{R}^3$ contains a cycle that is a nontrivial knot. Similarly, a graph is said to be *intrinsically linked* (IL) if every embedding of it in $\mathbb{R}^3$ contains a nontrivial link. Sachs [5] and Conway and Gordon [1] showed that K_6, the complete graph on six vertices, is IL. Conway and Gordon [1] also showed that K_7 is IK.

A ∇Y *move* on an abstract graph consists of removing the edges of a 3-cycle `abc` in the graph, and then adding a new vertex `v` and connecting it to each of the vertices `a`, `b`, and `c`, as shown in Fig. 1. The reverse of this move is called a Y∇ *move*. Note that in a Y∇ move, the vertex `v` cannot have degree greater than three.

Sachs [5] noticed that additional IL graphs can be obtained from K_6 by doing finite sequences of ∇Y and Y∇ moves on it. Motwani, Raghunathan, and Saran [3] showed that performing a ∇Y move on any IK or IL graph produces a graph with the same property. Robertson, Seymour, and Thomas [4] (Lemmas 1.2 and 5.1 (iii)) proved that a Y∇ move on any IL graph produces an IL graph again.

It has been an open question whether a Y∇ move on an IK graph always produces an IK graph again. We prove that the answer is negative, by giving a knotless embedding of a graph G_7 that is obtained from K_7 by ∇Y and Y∇ moves.

A graph H is a *minor* of another graph G if H can be obtained from G by a finite sequence of edge deletions and contractions and vertex deletions [2]. A graph is said to be *minor minimal* with respect to a property if the graph has that property but no minor of it has the property.

We work with connected, finite, simple graphs, i.e., graphs with no loops (an edge whose endpoints are the same) and no double-edges (two edges with the same pair

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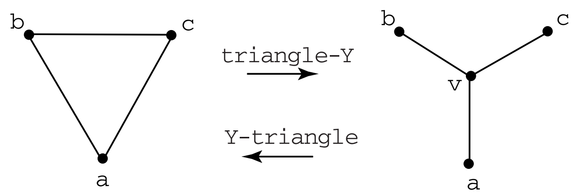

Fig. 1. ∇Y and Y∇ moves.

of endpoints). This is because loops and double-edges do not affect whether or not a graph is IK or IL: they can always be embedded such that they bound small disks with interiors disjoint from the rest of the graph. Thus, in edge contractions and Y∇ moves on an abstract graph, whenever a double-edge is introduced, one of the two edges is deleted.

2. Description of the graph G_7

We label the seven vertices of the abstract graph K_7 with the letters `a` through `g`. We perform the following five ∇Y and two Y∇ moves on $G_0 = K_7$ to obtain the graph G_7.

1. $G_0 \to G_1$ by ∇Y on `abc`, with new vertex `h` as center.
2. $G_1 \to G_2$ by ∇Y on `ade`, with new vertex `i` as center.
3. $G_2 \to G_3$ by ∇Y on `afg`, with new vertex `j` as center.
4. $G_3 \to G_4$ by ∇Y on `bdf`, with new vertex `k` as center.
5. $G_4 \to G_5$ by ∇Y on `beg`, with new vertex `l` as center.
6. $G_5 \to G_6$ by Y∇ on `hij`, deleting vertex `a`.
7. $G_6 \to G_7$ by Y∇ on `hkl`, deleting vertex `b`.

3. G_7 is not IK

Theorem 1. *The* Y∇ *move does not preserve intrinsic knottedness.*

Proof. Recall that K_7 is IK, ∇Y moves preserve IKness, and G_7 is obtained from K_7 by ∇Y and Y∇ moves. Thus it suffices to prove that the embedding of G_7 shown in Fig. 2 has no nontrivial knots.

Fig. 2 contains seven crossings, numbered 1–7. Note that rotating this diagram by 180° about a horizontal line through its center leaves the embedded graph invariant, swaps crossing 1 with 2, and 6 with 7, and leaves crossings 3, 4, 5 fixed. And rotating the diagram by 180° about a vertical line through the center also leaves the embedded graph invariant, but swaps crossing 1 with 7, 2 with 6, and 3 with 5.

Suppose towards contradiction that this embedded graph contains a nontrivial knot K. The proof consists of the following three steps.

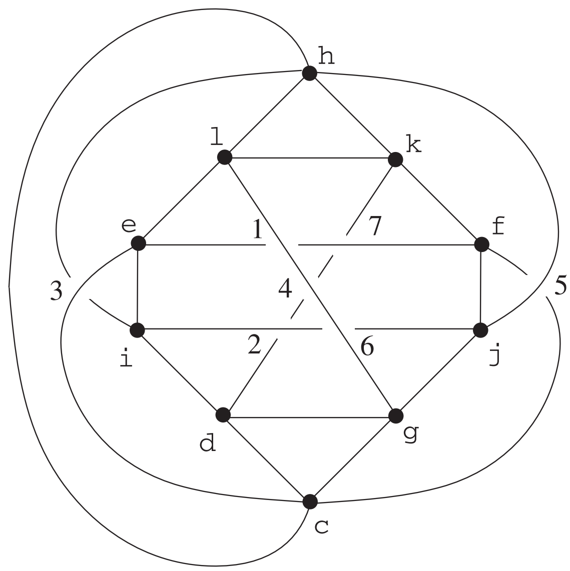

Fig. 2. A knotless embedding of G_7.

STEP 1. K must contain exactly one of the edges `ef` and `ij`.

Proof. We will show that if K contains neither or both edges, then it is a trivial knot.

Suppose K contains neither `ef` nor `ij`. Then it does not contain any of the crossings 1, 2, 6, and 7, and must therefore contain crossings 3, 4, and 5. Hence K contains the edges `ec`, `ih`, `dk`, `gl`, `fc`, and `jh`. If K contains `ei` or `fj`, then at least one of its crossings can be untwisted, making K trivial. So K must contain `el`, `fk`, `id`, and `jg`. Then K is easily seen to be trivial.

Now suppose K contains both `ef` and `ij`. Then it cannot contain both 3 and 5, since otherwise it would be a link. So, by symmetry, we can assume K does not contain 5. Furthermore, if K contains `fj`, then it is trivial. It follows that K must contain at least one of `fk` or `jg`. By symmetry, we can assume it contains `fk`. We claim that K must contain `dk`, since otherwise it will contain at most three crossings, 3, 1, and 6; but 1 and 6 do not alternate, which makes K trivial. Now, `dkfe` can be isotoped, with fixed endpoints, to eliminate 1, 4, and 7. So K must contain 3, 2, and 6. If K contains `jh`, 3 and 6 will not alternate, making K trivial. So K must contain `jg`. But then 6 can be isotoped away, again making K trivial. This proves Step 1.

So, by symmetry, we can assume K contains `ef` and not ij. Hence K does not contain crossings 2 or 6.

STEP 2. *K* must contain 1, 4, and 7.

Proof. Suppose, towards contradiction, that *K* does not contain `gl`. Then it contains at most three crossings, 3, 5, and 7; but `ec`, `cf`, and `fe` form a cycle, and therefore only links contain all three crossings 3, 5, and 7. Hence *K* contains `gl`. By a symmetric argument, *K* contains `dk`. Thus *K* contains crossings 1, 4, and 7.

STEP 3. *K* contains exactly one of 3 and 5.

Proof. If it contains both, it will be a link. If it contains neither, it will be trivial, since 1 and 4 do not alternate.

So, by symmetry, we can assume that *K* contains 1, 3, 4, and 7, and no other crossings. As *K* does not contain `ij`, this implies that *K* contains `di`. But `hidk` is isotopic, with fixed endpoints, to `hk`. Thus *K* is isotopic to a knot that contains only crossing 1, and therefore is trivial. □

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