1. INTRODUCTION

INTRODUCTION

FLAT SPACETIME

| Review of Special Relativity |
| Four-Vectors |
| Index Notation |

TENSORS

| Arbitrary Coordinates |
| Tensor Equations |
| Maxwell's Equations |
| Geodesics |

SCHWARZSCHILD BLACK HOLES

| The Schwarzschild Metric |
| Particle Orbits |
| Precession of the Perihelion |
| Photon Orbits |
| Deflection of Light |
| Event Horizon |
| Alternative Coordinates |
| Black Hole Thermodynamics |

THE CALCULUS OF CURVATURE

| The Absolute Gradient |
| Geodesic Deviation |
| The Riemann Tensor |

THE EINSTEIN EQUATION

| The Stress-Energy Tensor |
| The Einstein Equation |
| Interpreting the Equation |
| The Schwarzschild Solution |

COSMOLOGY

| The Universe Observed |
| A Metric for the Cosmos |
| Evolution of the Universe |
| Cosmic Implications |
| The Early Universe |
| CMB Fluctuations & Inflation |

GRAVITATIONAL WAVES

| Gauge Freedom |
| Detecting Gravitational Waves |
| Gravitational Wave Energy |
| Generating Gravitational Waves |
| Gravitational Wave Astronomy |

SPINNING BLACK HOLES

| Gravitomagnetism |
| The Kerr Metric |
| Kerr Particle Orbits |
| Ergoregion and Horizon |
| Negative-Energy Orbits |

this   depends on   this

[**How to Use This Chapter.** This chapter has a different structure than the remaining chapters in this text (see the Preface). When I teach from this text, I do *not* have my students read this chapter; instead, I present the material appearing in this chapter in the form of a 40-minute lecture on the first day of class. To support that lecture, I give the students a handout that provides an outline of the ideas and also the main figures that appear in this chapter. (The handout appears at the end of this chapter and also is available on the text's website). I find this to be an efficient way to use time in the first session (since my students have done no reading in advance) and also an effective way to get them excited about the course.

I have therefore provided this chapter mostly for instructors (either to help them prepare for a similar lecture or to provide a reading assignment if they prefer to do other things with the first class session) and for those using this book for self-instruction. So that all users have a similar experience, I have structured this chapter more or less as a transcript of the opening lecture I give to my students, rather than using the workbook structure found in all of the remaining chapters.]

Introduction. General relativity, at its heart, is very simple. While it is true that its mathematics is at times challenging and its interpretation can be mind-bending, the theory's core concepts are straightforward, plausible, and easy to understand. This simplicity is the core of the theory's great elegance and beauty, and sets a high standard that other modern physical theories struggle to emulate.

In what follows, I will provide in a few pages a complete overview of the theory's conceptual structure. The entire rest of this book contains little more than details about and applications of these core ideas!

The Curious Equality of Gravitational and Inertial Mass. Consider first two particles with charges Q and q interacting electrostatically with each other but nothing else. Coulomb's law and Newton's second law then imply that

$$\frac{kQq}{r^2} = \left(\frac{kQ}{r^2}\right)q = F_e = m_I a \tag{1.1}$$

where k is Coulomb's constant, r is the particles' separation, F_e is the magnitude of the electrostatic force that Q exerts on q, a is the magnitude of q's acceleration, and m_I is that particle's inertial mass, expressing how strongly it resists being accelerated by a given force. We can (and typically do) interpret the quantity in parentheses as being the magnitude of the electric field $\vec{E}$ that the particle with charge Q creates at the other particle's position, and the quantity q as describing how strongly that other particle responds to or "couples" to that field.

Now consider two particles with masses M and m_G interacting gravitationally with each other but nothing else. Newton's law of universal gravitation and Newton's second law then imply that

$$\frac{GMm_G}{r^2} = \left(\frac{GM}{r^2}\right)m_G = F_g = m_I a \tag{1.2}$$

where G is the universal gravitational constant, F_g is the magnitude of the gravitational force that M exerts on m_G, and the other quantities are as before. Guided by equation 1.1, we interpret the quantity in parentheses as being the magnitude of the gravitational field $\vec{g}$ that the particle with mass M creates at the other particle's position and m_G as expressing how strongly that other particle couples with that field.

Now, I have put different subscripts on m_G and m_I to emphasize that, though they both describe properties of the same particle, conceptually these properties are quite different. The quantity m_I expresses the particle's **inertial mass**, that is, how strongly it resists being accelerated by a given force, while m_G expresses its **gravitational mass**, that is, how strongly it couples to a gravitational field. These are *completely distinct physical quantities expressing completely different concepts*. We should no more expect m_G to be linked with m_I than we would q to to be linked with m_I in equation 1.1.

So, canceling m_G and m_I on both sides of equation 1.2 (as we all have been trained to do) makes the *very* big assumption that *an object's inertial mass is the same as its gravitational mass.* Most physicists before Einstein simply *assumed* that this was true. But is it really? We now know that a gold nucleus, for example, has a smaller inertial mass (as measured in a mass spectrometer) than an equal number of protons and neutrons because of the gold nucleus's large binding energy. Might the gold nucleus's gravitational mass depend on simply the number of nucleons and not on the binding energy? What is the experimental evidence?

To see how we might answer this question, divide both sides of equation 1.2 by m_I but *don't* assume that m_G and m_I cancel. We get for the acceleration

$$\left(\frac{GM}{r^2}\right)\left(\frac{m_G}{m_I}\right) = a \tag{1.3}$$

If m_G and m_I are not the same, then the ratio m_G/m_I could be different for different objects, which would imply that they would experience different accelerations in the same gravitational field. This is something that we can investigate experimentally.

Galileo and Newton both provided some basic evidence for the equality of m_G and m_I (to about one part in a thousand) during the 1600s, and this satisfied the community for a long time. However, the question began to interest physicists again in the late 1800s. Starting with a famous experiment performed by Eötvös in 1890, physicists during the 20th century have designed increasingly sophisticated and accurate experiments using a number of different techniques. Current experiments have established that m_I and m_G are equal to within at least one part in a billion in a wide variety of circumstances, and the most precise experiments to date (which use a sensitive torsion balance to look for differences in the acceleration of different objects objects in the sun's gravitational field; see the website www.npl.washington.edu/eotwash/ for details) yield uncertainties of a few parts in 10^{13}.

Now, the fact that these two seemingly distinct quantities are the same to almost 13 significant digits begs for explanation. General relativity provides a simple and elegant explanation.

The Geodesic Hypothesis. The first step toward this explanation involves recognizing that if m_G were really equal to m_I, then equation 1.3 would imply that all objects in a given gravitational field experience the same acceleration, and thus that *all objects would follow the same trajectory in a given gravitational field* if launched from the same position with the same initial velocity, even if they differ in mass and/or other characteristics. Note that such a statement is *not* true in electrostatics: objects with different charges follow *different* trajectories in a given electric field, even if their initial positions and velocities are the same. But in the gravitational case, it is as if the trajectory were determined by the space through which the objects move rather than by anything about the objects.

But how can empty space determine a trajectory? In the two-dimensional space represented by a flat piece of paper, there is a unique path between any two points that has the shortest pathlength: that path is a straight line. In the two-dimensional space corresponding to the surface of a globe, the analogous paths are "great circles." Indeed, in the two-dimensional space corresponding to the surface of any smooth convex three-dimensional object, we can find the shortest path between two points by stretching a string tightly between those points. In a general space, we call the paths that represent the shortest (more technically, the extremal) distance between two points a **geodesic**. A space's geometric characteristics therefore define unique geodesic paths in that space.[1]

---

1. Technically, if two points in a space are separated by a distances large compared to the scale over which the space's curvature becomes significant, one may be able to find more than one geodesic connecting the points. For example, the poles of a sphere can be connected by an infinite number of great circles. But if the points are separated by distances small compared to that scale, the geodesic between a given pair of points is unique. Let's assume this.

The **geodesic hypothesis** of general relativity asserts simply that

A free particle follows a geodesic in spacetime.

(where "a free particle" is one free of non-gravitational interactions). According to this hypothesis, a gravitational field shapes spacetime, which in turn specifies the geodesics that particles must follow.

The geodesic hypothesis makes sense only in *spacetime*, not in three-dimensional space. To see this, consider a thrown ball moving in a parabolic trajectory from point *A* to point *B* in the space near the earth's surface. But I could also fire a bullet from point *A* in such a way that it passes through point *B*: because of its greater speed, such a bullet would follow a much shallower parabola between the points (see figure 1.1). But the definition of a geodesic implies that there should be a *unique* geodesic between points *A* and *B*. Therefore the ball and bullet, even though both are "free," cannot both be following a geodesic, contrary to the hypothesis!

However, if the ball and bullet follow geodesics in *spacetime*, the apparent paradox evaporates. Figure 1.2 shows graphs of the ball's and bullet's trajectories in space *and* time. From this graph, we can draw two important conclusions. First, we see that even though the ball and bullet start out from point *A* at the same time (by hypothesis), they do not end up at point *B* at the same time, so they are *not* traveling between the same two points in *spacetime*. Two objects that do travel between between *A* and *B* in the *same time* would also have to have the same initial velocity and therefore *would* follow exactly the same trajectory in both space and time.

Second, though the paths of the ball and bullet are clearly *different* geodesic paths when plotted in spacetime, if we measure time in meters of light-travel time, then figure 1.2 shows that both have approximately the *same* radius of curvature (roughly 1 light-year = 1 ly). These different geodesics thus share a common curvature that they plausibly get from the spacetime around the earth.
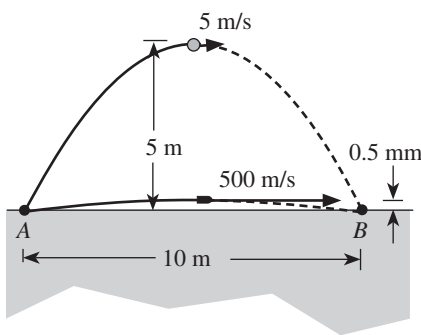

FIG. 1.1 The paths of a freely-falling bullet and a ball moving between points *A* and *B* in *space* near the earth are different: there is no unique path for a freely-falling object connecting two points in *space*. (Adapted from Misner, Thorne, and Wheeler, *Gravitation*, Freeman, 1973, p. 33.)
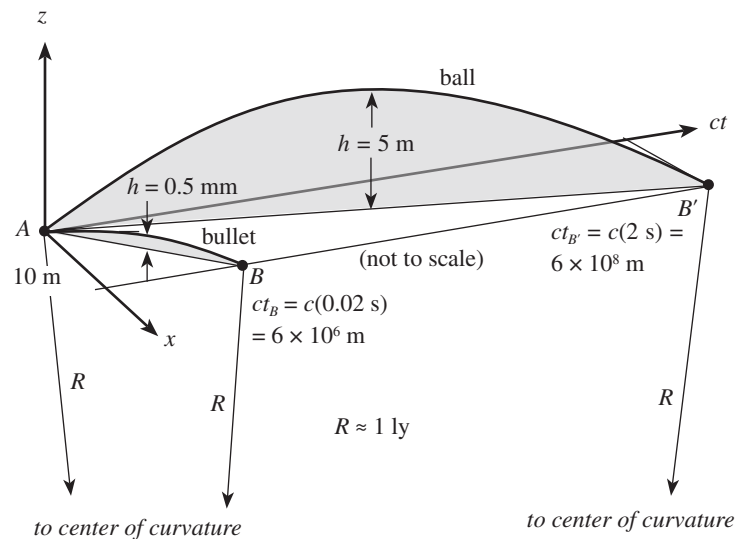
FIG 1.2 Plotted in *spacetime*, the bullet and ball paths have different ending times t_B and $t_{B'}$, so their final points *B* and *B′ in spacetime* are *not* the same. However, if we express displacements along the time axis of this spacetime graph in terms of *ct* (where *c* is the invariant speed of light) so that all axes have the same units of meters, then the two paths *do* turn out to have approximately the same radius of curvature $R \sim$ 1 light-year = 1 ly (see problem P1.1). Note that two projectiles moving between the same positions in space *and* time would also have to have the same initial *speed* and thus *would* follow the same (unique) path in spacetime. (Adapted from Misner, Thorne, and Wheeler, *Gravitation*, Freeman, 1973, p. 33.)

Why Gravitational Mass *Is* Inertial Mass. If we accept the geodesic hypothesis, then gravitational and inertial mass are the same thing, as I will now argue. Note that near the earth's surface, the geodesic for an object released from rest is a trajectory where the object accelerates downward at a rate of $g = 9.8$ m/s^2. According to the geodesic hypothesis, this is the "natural" path for a free object to follow, analogous to the straight-line geodesic an object would "naturally" follow in deep space (far from any gravitating objects). Now in deep space, accelerating an object away from a straight-line geodesic requires one to exert a force on the object. Analogously, if I hold an object at rest near the earth, I must exert an upward force on the object sufficient to accelerate it at a rate of $g = 9.8$ m/s^2 relative to the downward geodesic it naturally wants to follow. The magnitude of force required, according to Newton's second law, is simply $m_I g$, where m_I is the object's *inertial* mass.

However, it is precisely the magnitude of the *upward* force required to hold an object at rest that scales and balances measure when we "weigh" an object. In Newtonian mechanics, we imagine this upward force to be balanced by (and equal in magnitude to) a "gravitational force" $m_G g$ acting on the object, and thus we imagine the scale to register the object's "weight," which (after division by g) yields the object's gravitational mass m_G. But from the perspective of general relativity, the only real force acting on the object is the upward force (since a net force is required to accelerate an object relative to its geodesic), and that net force has a magnitude of $m_I g$. Therefore, when we *think* we are measuring an object's gravitational mass using a scale, we are *really* measuring its resistance to acceleration. So of course $m_G = m_I$: they are really the same thing!

Inertial and Noninertial Reference Frames. The paragraph above implies that in general relativity, we consider an object's "weight" (that is, the gravitational force acting on it) to be fictitious, not real. How can this be? To answer this question, we have to rethink the definition of inertial and noninertial reference frames.

In Newtonian mechanics, we typically define an inertial reference frame (IRF) to be "a frame in which a free object initially at rest remains at rest." However, it seems that we immediately waive this definition when we treat a reference frame at rest on the earth's surface as being even approximately inertial, since a free object initially at rest obviously does *not* remain at rest, but rather accelerates downward at a rate of g! The Newtonian explanation, of course, is that an object near the earth's surface is not "free," but rather subject to a gravitational force exerted on it by the earth, and that is why it accelerates. However, the only evidence for this "force" is the observed acceleration of a dropped object, which is unnatural only if we *assume* that a reference frame at rest on the earth's surface is inertial.

In general relativity, we take the definition of an IRF as given above literally and seriously. A reference frame at rest on the earth's surface is therefore *not* inertial, since a free object does not remain at rest. The only reference frames near the earth that are even approximately inertial are *freely falling frames*. We know, for example, that in a freely falling frame such as an orbiting space shuttle, an object placed at rest in midair remains floating at rest, consistent with the definition of an IRF!

Of course a Newtonian physicist would claim that this is an illusion, because both the shuttle and object happen to fall toward the earth with the same acceleration, and so remain at rest with each other. So is the decision about whether to take a frame at rest on the earth's surface or a freely falling frame as being an IRF merely a matter of perspective? No! One of Einstein's greatest triumphs was to show that this choice has physical consequences that we can examine experimentally.

The Equivalence Principle. Einstein pointed out that if a freely falling reference frame is truly an IRF, then it should be *physically equivalent* to a freely floating frame in deep space (far from any massive bodies), in the sense that any experiment performed in a freely falling frame should yield the same result as in the deep-space frame.

(a)

accelerating frame

photon

floating IRF

(b)

accelerating frame
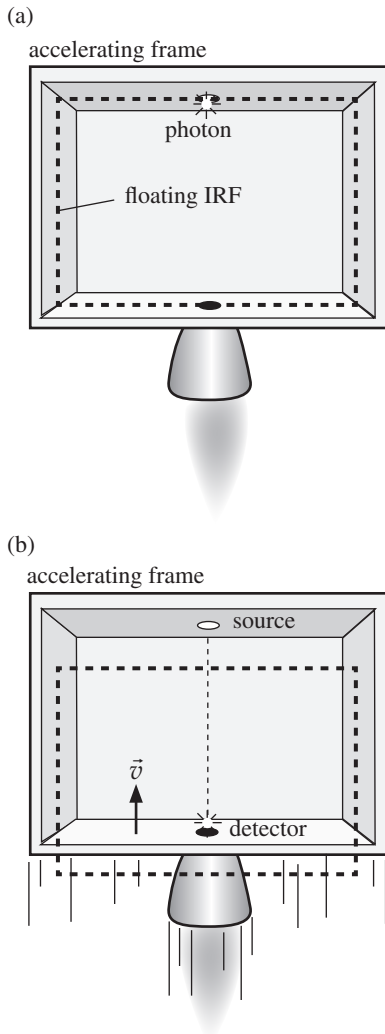
source

$\vec{v}$

detector

FIG. 1.3 (a) At the instant that an accelerating frame in deep space is at rest with respect to a floating inertial frame (IRF) in deep space, a photon is emitted by a source in the accelerating frame's ceiling. (b) By the time that the photon reaches the detector on the floor, the accelerating frame (and thus the detector) is moving upward relative to the IRF. The detector will therefore measure the photon's frequency to be blue-shifted. Since an accelerating frame in deep space is physically equivalent to a frame at rest on the earth's surface, we would expect to see the photon blue-shifted in a frame on the earth as well.

Similarly, a frame at rest near the earth's surface, since it is accelerating upward at a rate of g = 9.8 m/s^2 with respect to the inertial frames in its vicinity, should be *physically equivalent* to a rocket-powered noninertial frame accelerating at a rate of g relative to freely floating frames in deep space. This is one statement of what physicists call Einstein's **Equivalence Principle**.

Consider the following experimental test of this principle. Imagine that light of a certain frequency is emitted by a source attached to the ceiling of a frame at rest on the surface of the earth. This light is detected, and its frequency measured, by a detector on the floor of that frame. If such a frame is really *inertial* (in contradiction to general relativity), then the detected light will be observed to have the same frequency as the emitted light. If such a frame is *noninertial* (as required by the Equivalence Principle), on the other hand, then the light will be observed to be slightly blue-shifted, as I will now argue.

The Equivalence Principle requires that what we observe in a frame at rest near the earth's surface should be the same as what we observe in an accelerating frame in deep space. Imagine that such an accelerating frame happens to be at rest with respect to a floating IRF at the time (t = 0) that a certain photon is emitted by the accelerating frame's ceiling source (see figure 1.3). In the floating IRF, that photon's frequency remains constant. But by the time t' the photon reaches the detector on the floor of the accelerating frame, that accelerating frame is moving upward relative to the IRF at a speed of $v = gt'$. Therefore, in the floating IRF, the detector at the time of reception is moving with this speed toward the source (which was at rest in the IRF at the time of emission). Therefore, the detector will observe the light to be slightly Doppler-shifted toward the blue. This shift will be very small (because t' will be small: see problem P1.2 for a calculation of the magnitude), but nonzero.

The effect is indeed so small that it eluded firm experimental verification for more than 50 years after Einstein first predicted it. However, in 1959, R. V. Pound and G. A. Rebka were able to verify this effect in the reference frame corresponding to a 22.5-m high tower in the Jefferson Physical Laboratory at Harvard University (see Pound and Rebka, "Gravitational Redshift in Nuclear Resonance" *Phys. Rev. Lett.* **3**, 439–441, 1959). This experiment used gamma rays from a radioactive sample of Fe-57 as a source, and took advantage of the Mössbauer effect to make a very precise measurement of the frequency shift in the gamma rays as they were absorbed by an Fe-57 sample at the other end of the tower. This experiment verified the predicted blue-shift to an uncertainty of about 10%. Subsequent experiments have verified this in earth-based frames to within one part in 10^4.

We see that experimental evidence firmly supports the conclusion that frames at rest on the earth's surface are really noninertial, and consequently that freely falling frames *are* inertial. This means that the "force of gravity" that *appears* to press us to the floor in a frame at rest near the earth is really a fictitious force, as fictitious as the force that appears to press us to the floor in a frame that is accelerating upward. Indeed, this force vanishes in a truly inertial (that is, freely falling) frame: in such a frame, objects are exposed as being truly "weightless." Any force that appears or disappears depending on our choice of reference frame cannot be real.

The Reality of Gravity. So is gravity *entirely* fictitious? Is there *nothing* about gravity that is real (that is, observable in an inertial frame)? The answer to both of these questions is no. Gravity *is* real, and *does* have an aspect observable in an inertial frame, it is just not the downward force we usually consider "gravity."

To see what this aspect is, consider a large room freely falling toward the earth. Imagine that we place four balls so that they float initially at rest with respect to the room; one near the room's ceiling, one near its floor, one near a wall, and one near the opposite wall (see figure 1.4). What happens to these balls as the room falls?

(a)

freely-
falling
frame
near the
earth

(b)

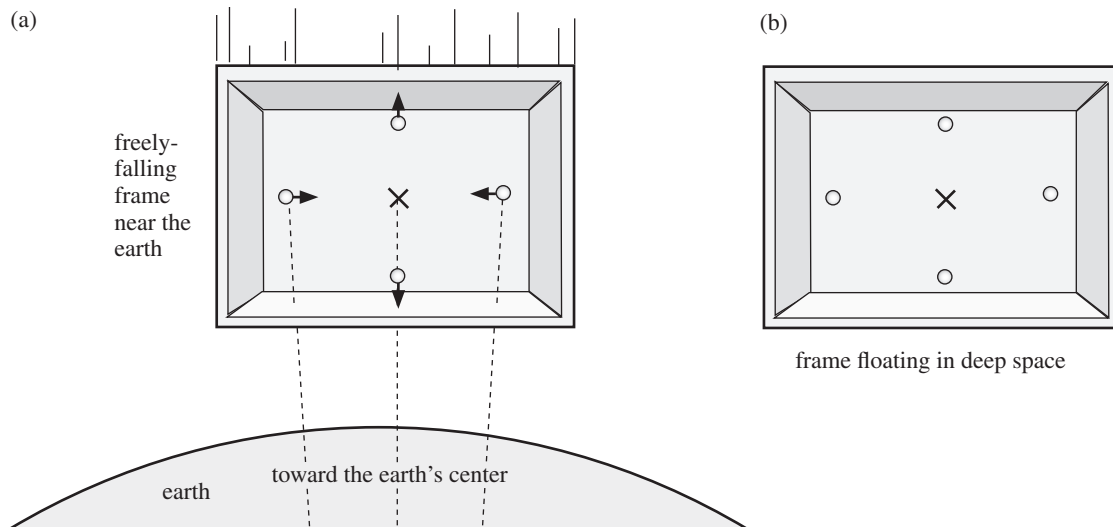frame floating in deep space

earth

toward the earth's center

FIG. 1.4 (a) Because the gravitational field of the earth (indeed, any gravitating object) is non-uniform, off-center floating balls in a freely falling frame will experience small accelerations relative to the frame's center of mass. (b) Such accelerations are *not* observed in a frame floating in deep space: floating balls initially at rest remain truly at rest.

To predict what happens, let us retreat for the moment back into the Newtonian picture (which will predict the correct behavior even if it does not provide the correct interpretation). In that picture, the room's center of mass falls toward the earth with a certain acceleration. The ball near the ceiling is just a bit farther from the earth's center than the room's center of mass, so it experiences a slightly smaller acceleration, just as the ball near the floor experiences a slightly larger acceleration. The balls near the walls accelerate toward the earth's *center* and thus along lines angled slightly inward with respect to the direction along which the room accelerates. So as time passes, we will observe the top and bottom balls to accelerate *away* from the room's center, while the side balls will accelerate *toward* the room's center. This is *not* the behavior that we would observe in a frame floating in deep space: in such a frame, the balls would remain strictly at rest.

The relative accelerations of off-center free bodies, then, is something we *do* observe in an inertial (freely falling) frame near a gravitating object, but *not* in an inertial frame in deep space. These relative accelerations therefore represent a frame-independent (and thus real) indication that we must be near a gravitating object.

We call this aspect of gravity the **tidal effect** of a gravitational field, because (as Newton himself first realized) this effect explains tides on the earth's surface. Note that we can consider the earth to be a frame freely falling in the moon's gravitational field. Like the balls in our freely falling room, ocean waters on the sides of the earth closest to and farthest from the moon will accelerate away from the earth's center and thus bulge outward, while ocean waters on the sides will press inward. This explains the 12-hour tidal variation of the depth of the ocean.

Spacetime Is Curved. How do we interpret these tidal effects from the perspective of general relativity? Figure 1.5 shows a spacetime graph of the trajectories of the two side balls in our falling-room experiment. Since these balls are initially at rest with respect to each other, their paths in spacetime are initially parallel (they remain at an initially constant separation as time passes). As time progresses, however, they eventually begin to move toward each other with increasing speed, so the paths curve toward each other as shown.
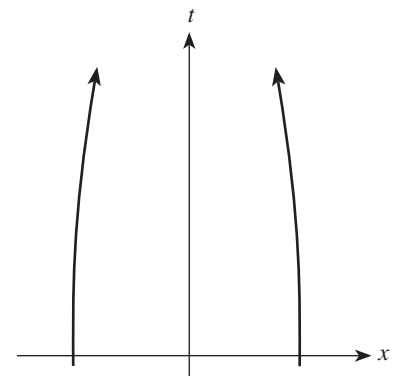
FIG. 1.5 Plotted in spacetime, the geodesics that the balls in figure 1.4 follow (as measured in the freely-falling frame) are initially parallel (the balls have initially constant separation), but gradually bend toward each other (because their separation eventually decreases). This bending of initial parallel lines signals that the underlying spacetime is *curved*.

But remember that these paths are *geodesics* (that is, the straightest possible lines) in spacetime. A fundamental axiom of Euclidean (flat plane) geometry is that initially parallel straight lines remain parallel. Here we see that initially parallel geodesics in spacetime (the straightest possible lines we have in spacetime) do not *remain* parallel. While this violates Euclid's axiom for plane geometry, this behavior is typical of curved space. For example, on the two-dimensional space corresponding to the surface of a globe, lines of longitude are great circles (geodesics). These lines are parallel at the equator but do not remain parallel as one goes toward the poles. We conclude that the relative acceleration of geodesics near a gravitating object indicates that the geometry of spacetime is *curved* (non-Euclidean) there. This curvature of spacetime is the frame-independent signal that a gravitational field is present. Moreover, once we understand exactly *how* spacetime is curved near a gravitating body, we can calculate that spacetime's geodesics and therefore predict how freely-falling bodies will move.

The Einstein Equation. Given the geodesic hypothesis, then, the central task of a theory of gravity is to predict how a gravitating body affects the curvature of spacetime. On November 25, 1915, Einstein completed the theory of general relativity by proposing an equation that linked the presence of matter and energy to the curvature of spacetime, an equation that we now call the **Einstein equation** (or the Einstein Field Equation). This equation reads as follows:

$$G^{\mu\nu} = 8\pi G T^{\mu\nu} \tag{1.4}$$

where $G^{\mu\nu}$ is a 4×4 matrix (tensor) that describes the curvature of spacetime at a given point in space and time, G is the universal gravitational constant, and $T^{\mu\nu}$ is a 4×4 matrix describing the density and flow of matter and energy at the same point in space and time. This equation and the **geodesic equation** used to calculate geodesics in an arbitrarily curved spacetime comprise the core equations of general relativity.

General Relativity in a Nutshell. In the chapters that follow, we will explore in great detail the mathematical meaning of both the Einstein equation and the geodesic equation. For our purposes at the moment, however, it is enough to understand their *physical* meaning. To summarize, if we know how spacetime is curved, we can use the geodesic equation (the mathematical equivalent of stretching a string between two points in spacetime) to calculate how objects will move in that spacetime. If we know the density and flow of matter and energy in spacetime, we can use the Einstein equation to calculate how spacetime is curved. The great physicist John Archibald Wheeler summarized the theory's essence even more briefly this way:

Spacetime tells matter how to move; matter tells spacetime how to curve.

This is general relativity in a nutshell. What could be simpler? Our task in what follows will simply involve unfolding the implications of this simple but profound statement.
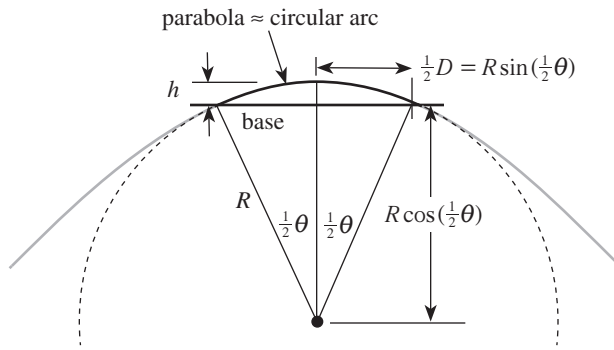
HOMEWORK PROBLEMS

FIG. 1.6 This figure illustrates how we can model a parabola with height h and base D by a circular arc having an appropriate radius R.

P1.1 This problem explores the claims made in figure 1.2.
a. Show that the values of h and t for the bullet and ball trajectories in that figure are consistent with a gravitational acceleration of $g = 10$ m/s^2.
b. The ball's and bullet's paths in the spacetime graph in figure 1.2 are actually *very* gently curved parabolas when you consider how long their bases are in that diagram. As shown in figure 1.6, we can use a circular arc as an excellent approximation for a gently curved parabola. We can calculate that arc's effective radius R, and thus quantify the path's curvature in a direct and reasonably intuitive way. Note from the figure that the peak of a circular arc that spans an angle θ is a distance R away from the circle's center, but the center of the arc's base is only $R\cos(\tfrac{1}{2}\theta)$ from the center. The height of the arc's peak above its base is therefore $h = R[1 - \cos(\tfrac{1}{2}\theta)]$. Since the angle θ will be quite small in this case, we can expand $\cos(\tfrac{1}{2}\theta)$ in a power series and drop higher order terms: $\cos(\tfrac{1}{2}\theta) \approx 1 - \tfrac{1}{2}(\tfrac{1}{2}\theta)^2 = 1 - \tfrac{1}{8}\theta^2$ (assuming that θ is in radians and is much smaller than 1). In the same limit, $\sin(\tfrac{1}{2}\theta) \approx \tfrac{1}{2}\theta$, so the base's length D is thus $2R\sin(\tfrac{1}{2}\theta) \approx R\theta$. Now, in figure 1.2, the length of each curve's base in the spacetime diagram is almost exactly equal to ct, where t is the time required for the projectile to go between spatial points A and B. Combine the approximations above with the given values of h and t for each path to show that effective radii of curvature of the ball's and bullet's paths in figure 1.2 are both approximately $R \approx 10^{16}$ m ≈ 1 ly. Also check that θ is indeed very small for both paths, justifying the approximations we have made.

P1.2 Imagine that a laser on the ceiling of a laboratory on the earth emits a flash of light directed toward a sensor on the floor a distance $d = 25$ m below (the laboratory is in a tower). This lab is equivalent (from the point of view of the gross effects of gravity) to an identical lab accelerating upward in deep space with a uniform acceleration of magnitude g. Imagine that we observe the flash being emitted and detected in an inertial lab surrounding the accelerating lab. For the sake of simplicity, imagine that the two labs are at rest with respect to each other at the instant the flash is emitted. In the time it takes the flash to reach the floor (as measured in the inertial lab), the accelerating lab attains a certain speed v relative to the inertial lab. Thus (according to observers in the inertial lab) the floor detector in the accelerated lab is moving toward the laser with speed v at the time the pulse is detected, so the floor detector measures the laser light's wavelength to be blue-shifted to the value λ given by the relativistic Doppler shift formula $\lambda/\lambda_0 = \sqrt{(1 - v/c)/(1 + v/c)}$, where λ_0 is the wavelength of the light as emitted by the laser and v is the detector's speed relative to the laser at the time of detection.
a. Argue that the fractional shift in wavelength is

$$\frac{\lambda_0 - \lambda}{\lambda_0} \approx \frac{gd}{c^2} \qquad (1.1)$$

when $gd/c^2 \ll 1$ and $v/c \ll 1$. (*Hint:* You will find the binomial approximation $(1 + x)^n \approx 1 + nx$ helpful. This approximation is accurate to order x^2.)
b. What would be the fractional shift in wavelength in a lab on the earth's surface?
c. What would be the fractional shift in wavelength if the lab were located on the surface of a neutron star having a mass of $M = 3.0 \times 10^{30}$ kg (≈ 1.5 the mass of the sun) and a radius of $R = 12$ km? [*Hint:* First estimate the magnitude of $\vec{g}$ using Newton's law of universal gravitation. You can find the value of the universal gravitational constant G on the inside front cover.]

P1.3 Another consequence of the Equivalence Principle is that light will be bent in a gravitational field. This has never been measured on the surface of the earth, but was verified qualitatively by observing starlight passing near the sun's edge during a total eclipse in 1919. Why can't this experiment be done on the earth's surface? Let's predict how much bending we should see in a laboratory at rest on the earth's surface. What one would observe in such a laboratory should be the same as what one would observe in a laboratory accelerating in deep space with a uniform acceleration of $\vec{a} = -\vec{g}$, where $\vec{g}$ is the local acceleration of gravity on the earth's surface. Imagine that a laser at one end of the laboratory emits a beam of light that originally travels parallel to the laboratory floor.

P1.3 (continued)

This light shines on the opposite wall of the laboratory a horizontal distance $d = 3.0$ m away from the laser.

a. Find the magnitude of this deflection in a laboratory on the surface of the earth.

b. Find the magnitude of this deflection if the laboratory sits on the surface of a neutron star having a mass of $M = 3.0 \times 10^{30}$ kg (≈ 1.5 the mass of the sun) and a radius of $R = 12$ km. [*Hint:* First estimate the magnitude of $\vec{g}$ using Newton's law of universal gravitation. You can find the value of the universal gravitational constant G on the inside front cover.]

P1.4 We can calculate how much starlight passing the sun's edge will be deflected (according to the Equivalence Principle) as follows. Assume that the light's deflection is so small that we can approximate a given photon's trajectory by a straight line along the x axis that grazes the sun's surface, as shown in figure 1.7. Assume that the photon has the same acceleration as any other object as it passes near the sun and that this acceleration has the magnitude $|\vec{a}| = GM/r^2$ predicted by Newtonian physics (where $G = 6.67 \times 10^{-11}$ N·m²/kg², M is the sun's mass, and r is the distance between the photon and the sun's center), but that the photon's speed remains $\approx c$ (the speed of light) during the whole process. As figure 1.7 shows, the sine of the deflection angle δ will be equal to v_y/c, where v_y is the y component of the photon's final velocity $\vec{v}$. We can determine v_y by integrating $a_y\, dt$ over the entire trajectory, which you can do by expressing a_y as a function of x and R, writing $dt = dx/c$, and integrating from $x = -\infty$ to $+\infty$. Look up the resulting integral and note that for small angles, $\sin\delta \approx \delta$ (when δ is expressed in radians). Show therefore that the predicted deflection is $\delta = 4.2 \times 10^{-6}$ rad $= 0.87$ seconds of arc (which is small enough to justify the various approximations we have made above).
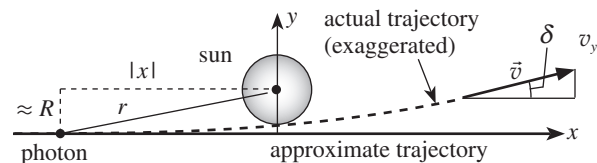

FIG. 1.7 This figure illustrates the trajectory of a photon passing near the sun's edge. Its deflection has been hugely exaggerated. The sun's radius R is about 700,000 km.

[*Note:* Einstein published this prediction in 1907 after first proposing the Equivalence Principle. However, as we will see in chapter 13, the full theory of general relativity actually predicts *twice* this deflection, as Einstein noted in 1915, and the latter result was verified during the 1919 eclipse. It turns out that while the Equivalence Principle works well in reference frames that are small enough compared to the scale over which any enclosed gravitational field varies significantly, in the calculation above we are implicitly using a frame that is large compared to that scale. In such a case, as Einstein himself found, a naive application of the Equivalence Principle yields incorrect results.]

P1.5 Imagine a freely falling reference frame near the surface of the earth. This reference frame has the form of a cube 44 m on a side. Imagine that floating balls are placed at point A in the frame's center, at point B 22 meters above A, and at point C 22 m below A. The frame's center of mass will fall at the same rate as the ball at A which is located at the frame's center. But due to tidal effects, the balls at B and C will fall a bit slower and faster respectively than the frame as a whole. What are the magnitudes of the accelerations of the balls at B and C *relative* to A (that is, what are the magnitudes of $\vec{a}_B - \vec{a}_A$ and $\vec{a}_C - \vec{a}_A$)? Feel free to use Newtonian physics in this calculation: we will show later that general relativity leads to the same results near the surface of the earth to many decimal places. (*Hint:* You will need to use the binomial approximation $(1 + x)^n \approx 1 + nx$. If you do not, you will find that your calculator does not keep enough digits to yield an accurate result.)

P1.6 Many smart phones (such as the iPhone) have built-in three-axis accelerometers. These accelerometers typically measure the x, y, and z force components per unit mass that must be applied to an internal proof mass to hold it at rest with respect to the rest of the phone. Now, if the phone is freely falling, it represents an inertial reference frame within which the internal proof mass *should* float without requiring *any* forces to hold it at rest. Find an app for your smartphone that is capable of logging acceleration data for all three axes, run the app, and then throw your phone in a nice parabola onto a suitable soft surface. (Note that *you* are responsible for any mishaps!) Use your collected data to argue that the phone is indeed an inertial reference frame during the time interval between leaving your hand and landing on the soft surface.

Two-sided handout for a lecture based on this material $\longrightarrow$

GENERAL RELATIVITY IN A NUTSHELL

I. FUNDAMENTAL IDEAS
   A. The curious equality of gravitational mass and inertial mass
      1. Illustrating the distinction:
         (a) Compare Coulomb's law with Newton's law of universal gravitation.
         (b) Note how the m's on either side of $GMm_G/r^2 = m_I a$ express *different* things.
      2. Experimental tests show that inertial mass = gravitational mass to 13 decimal places.
      3. It is not credible that this is an accident!
   B. The Geodesic Hypothesis
      1. A plausibility argument: the path followed by a falling object (since it is independent of the object's properties) seems to be a property of the *space*, not the object.
      2. What is a *geodesic*, anyway?
         (a) It is the straightest possible path (or path of "extremal" length) through a space.
         (b) The geometry of the space uniquely specifies such paths.
      3. A statement of the hypothesis: *A free particle follows a geodesic in spacetime.*
      4. Note: This works only if the paths are geodesics in *spacetime* (see figures 1.1 and 1.2).

II. IMPLICATIONS
   A. "Weight" really expresses an object's resistance to acceleration (relative to its geodesic)!
      1. An object's geodesic near the earth accelerates downward at $g = GM/r^2 = 9.8$ m/s^2.
      2. To hold an object at rest, we give it an upward acceleration g relative to its geodesic.
      3. This requires an upward force of $m_I g = m_I GM/r^2$.
      4. So the m's on either side of $GMm_G/r^2 = m_I a$ really express the same thing after all!
   B. Inertial reference frames (IRFs) and freely falling reference frames
      1. Definition of an IRF: a free object at rest (in the frame) remains at rest
      2. Near a gravitating object, only freely falling frames are genuine IRFs.
   C. The Equivalence Principle
      1. A frame on earth's surface is analogous to an accelerating frame in deep space.
      2. Implications:
         (a) bending of light in a gravitational field (see problem P1.3)
         (b) gravitational blue-shift (see figure 1.3)

III. THE REALITY OF GRAVITY
   A. What is real (that is, frame-independent) about gravity?
      1. Is it totally fictitious like "centrifugal force"?
      2. No! *Tidal* effects of gravity cannot be erased by change of reference frame (figure 1.4).
      3. So initially parallel paths of falling objects do not *remain* parallel (figure 1.5).
      4. Implication: *spacetime must be curved* (i.e., have a non-Euclidean geometry).
   B. The Einstein equation
      1. $G^{\mu\nu} = 8\pi G T^{\mu\nu}$, where
         (a) $G^{\mu\nu} = 4 \times 4$ symmetric tensor describing curvature of spacetime at a point
         (b) $T^{\mu\nu} = 4 \times 4$ symmetric tensor describing the density of mass-energy at that point
      2. This is the link between curvature of spacetime and mass that causes it.

IV. GENERAL RELATIVITY IN A NUTSHELL
Spacetime tells matter how to move, matter tells spacetime how to curve (J. A. Wheeler)
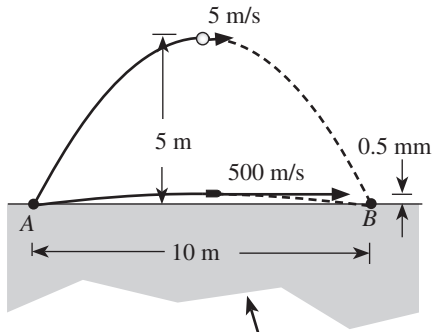
5 m/s

5 m

0.5 mm

500 m/s

A

B

10 m

FIG. 1.1

Adapted from Misner, Thorne, and Wheeler,
Gravitation, Freeman, 1973, p. 33.

z

ball

$h = 5$ m

$h = 0.5$ mm

A

bullet

ct

B'

$ct_{B'} = c(2$ s$) =$
6×10^8 m

(not to scale)

10 m

x

B

$ct_B = c(0.02$ s$)$
$= 6 \times 10^6$ m

R

R

R

$R \approx 1$ ly

to center of curvature

to center of curvature

FIG. 1.2

accelerating frame

(a)

photon

floating IRF

FIG. 1.3

accelerating frame

(b)

source

$\vec{v}$

detector

(a)

freely-
falling
frame
near the
earth

$\times$

(b)

$\times$

frame floating in deep space

t

x

FIG. 1.5

earth

toward earth's center

FIG. 1.4