Box 27.5

How to Generate a Numerical Solution for Equation 27.18

To find a numerical solution for $da/d\eta = [\Omega_m a + \Omega_v a^4]^{1/2}$ (equation 27.18), first divide up the η number line into equal steps of small $\Delta\eta$ starting at $\eta = 0$ and set up a table to store a value of a after each step. Then write equation 27.18 using a centered difference as an approximation for the derivative:

$$\frac{a_{i+1} - a_{i-1}}{2\Delta\eta} = \sqrt{\Omega_m a_i + \Omega_v a_i^4} \quad \Rightarrow \quad a_{i+1} = a_{i-1} + 2\Delta\eta\sqrt{\Omega_m a_i + \Omega_v a_i^4} \qquad (27.20)$$

where a_i is the value of a at time η_i. So, given the values of a_i and a_{i-1}, one can use this to calculate a_{i+1}. We need only two values $a(0)$ and $a_1 \equiv a(\Delta\eta)$ to start the calculation. Since the universe starts at infinite density, we will assume that $a(0) = 0$. At very small values of a, the matter term $\Omega_m a$ in equation 27.20 dominates, and the value of a halfway between 0 and a_1 is approximately $\frac{1}{2}a_1$, so you can calculate a_1 by setting up the difference equation 27.20 to span the step between $a(0)$ and a_1 (this step is only $\Delta\eta$ wide, not $2\Delta\eta$) and solving for a_1 to get $a_1 \approx \frac{1}{2}\Delta\eta^2$.

Exercise 27.5.1. Verify this last result.

HOMEWORK PROBLEMS

P27.1 What is the present physical distance d_{horiz} to the matter that created CMB we see now, according to the present best-fit model of the universe? Compare your result to $t_0 = 13.73$ Gy, the distance the light has actually traveled. (*Hint:* Use figure 27.3*a*.)

P27.2 Suppose that we discover that a certain kind of galaxy has a fixed linear size D. One could then compute the effective distance d_A to such a galaxy by dividing its size D by its measured angular size as viewed from earth. Find a formula for d_A in terms of χ that is analogous to equation 27.17 for the luminosity distance. Do not assume that the universe is flat.

P27.3 Using the method described in box 27.5, set up a spreadsheet, a Mathcad program, or some other kind of computer program to create a graph of $a(t)$ for our universe. Compare to figure 27.3*b*.

P27.4 Using the spreadsheet or program you created for problem P27.3 or a spreadsheet supplied by your instructor, create a graph of d_L versus z for our universe.

P27.5 A galaxy's measured redshift is $z = 1$. What is that galaxy's luminosity distance according to our current model of the universe? (*Hint:* Use figure 27.3*a*.)

P27.6 Quantum field theory provides a nice potential explanation for the vacuum energy density. The vacuum could very well (indeed, really should) have a zero point energy, much like a quantum harmonic oscillator or particle-in-a-box has nonzero energy even in the ground state. We can estimate what the energy density should be using dimensional analysis. Since this energy density should involve quantum mechanics, gravity, and relativity, it makes sense that it should be within an order of magnitude or so of whatever combination of $\hbar$, G, and c that has units of mass density.

a. Show that the Planck mass density $\rho_{Pl} = c^5/\hbar G^2$ is the only combination of these three constants that has the correct (SI) units.

b. Evaluate the Planck mass density and compare to the value of ρ_v that follows from the observed value of Ω_v and the definition of the same. Do you see a problem here? (This mystery remains unsolved.)

P27.7 Assume that we now observe galaxies to have redshift z if they emitted light at a certain cosmic time t. Show that the temperature T of the CMB at that t is

$$T = T_0(1 + z) \qquad (27.21)$$

where T_0 is the temperature of the cosmic microwave background now. (*Hint:* Consider equation 26.10.)