Thesis Outline:

Title: Fractal Waveforms Generated by Irrational Rotations
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1. Abstract *Usually written first, and then rewritten last.*

2. Introduction
   Why would anyone be interested in fractal waveforms? They have interesting perceptual and acoustic properties. For example, "Shepard's Tone" is a fractal waveform which gives the psycho-acoustic illusion of a perpetually descending tone. In this thesis, a process for constructing a fractal waveform will be described which generates a unique waveform for every irrational number. Fractal waveforms are pretty unusual in and of themselves, but *hearing* an irrational number is definitely very cool.

3. Backround
   J. Harrison constructed fractal closed curves from irrational rotations in an effort to provide counter-examples to Seifert's conjecture. This section will contain a bit about Siefert's conjecture, and why Harrison's counter-examples might work. The analysis of the curves themselves is based on the continued fraction expansion of the irrational number. Most of this thesis is devoted to explaining this analysis, and then applying it to a variation of Harrison's construction. This variation produces waveforms, i.e. functions of time, rather than closed curves in the plane.

4. **Definitions, Facts, and Theorems to be Presented**

   (a) Describe an irrational rotation of the circle. Show that any orbit of an irrational rotation is dense in S^1.

   (b) For any irrational number, α, the *continued fraction expansion of* α is given by:
   $$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$$
   where the a_i are strictly positive integers. The following horizontal notation is equivalent, and sometimes easier to read (and to typeset!):
   $$\alpha = [a_0, a_1, a_2, a_3, \ldots]$$
   The a_i are called the *partial quotients* of α.

   (c) *Fact:* The continued fraction expansion of α is infinite if and only if α is irrational.

(d) *Def:* The sequence of fractions:

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\cdots}{a_n}}}$$

is called the sequence of *convergents* of α. These convergents give successively better rational approximations to the irrational number α.

(e) *Fact:* The numerators and denominators of the convergents satisfy:

$$p_0 = a_0, \quad p_1 = a_1 a_0 + 1, \quad q_0 = 1, q_1 = a_1$$

$$p_n = a_n p_{n-1} + p_{n-2} \quad q_n = a_n q_{n-1} + q_{n-2}$$

(f) *Def:* Define yet another sequence of numbers:

$$r_0 = 1, r_1 = \alpha, r_{k+1} = r_{k-1} - a_k r_k$$

(g) *Fact:* After $(q_n + q_{n-1} - 1)$ rotations, R_α, the intervals between orbit points on S have length r_n or r_{n+1}.

(h) *Lagrange's Theorem:* Any quadrateic irrational, α has a continued fraction expansion which is periodic from some pint onward. Furthermore, any continued fraction with this property is that of a quadratic irrational.

(i) *Def:* The *Hausdorff s-dimensional measure* of a set, E is defined by:

$$H^s(E) = \lim_{\delta \to 0} \inf\{\sum_{i=1}^{\infty} |U_i|^s\}$$

where $\{U_i\}$ is any δ-cover of E.

(j) *Def:* The *Hausdorff-dimension* of E, $HD(E)$ is the unique value, d, such that $H^s(E) = \infty$ if $0 \leq s \leq d$, and $H^s(E) = 0$ if $d < s < \infty$.

5. Descriptions of the fractal waveforms generated by the algorithm. Prove that the waveforms are, indeed, fractals, i.e. that their Hausdorff dimension is not an integer.

6. Conclusions. What properties do these waveforms have? What do the acoustical properties of the waveforms tell us about the continued fraction expansion of the number?