NUMERICAL SOLUTION OF ODE IVPs

Overview

1. Quick review of direction fields.
2. A reminder about ________________$_{(1)}$ and ________________$_{(2)}$.
3. Important test: Is the ODE initial value problem ________________$_{(3)}$?
4. Fundamental concepts: Euler's Method.
5. Fundamental concepts: Truncation error.
6. Fundamental concepts: ________________$_{(4)}$ of a method.
7. Fundamental concepts: ________________$_{(5)}$ of a method.
8. Stiff ODEs.
9. Other methods overview.
10. Systems and higher order IVPs.
11. Solving IVPs with packaged software.

Numerical Solution of ODE IVPs

Direction Field Review

General First Order Ordinary Differential Equation:

$$y' = f(t, y)$$

- y' is shorthand for ____________________$_{(1)}$.
- $f(t, y)$ is a function of the ____________________$_{(2)}$ variable t and the ____________________$_{(3)}$ variable y.

Assumptions:

- $f(t, y)$ is defined and single valued in some rectangular region R in the $t - y$ plane.
- If $y = y(t)$ is a solution, then it is differentiable at all points in R. This allows us to plot a smooth curve.

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Direction Field Review – Demo

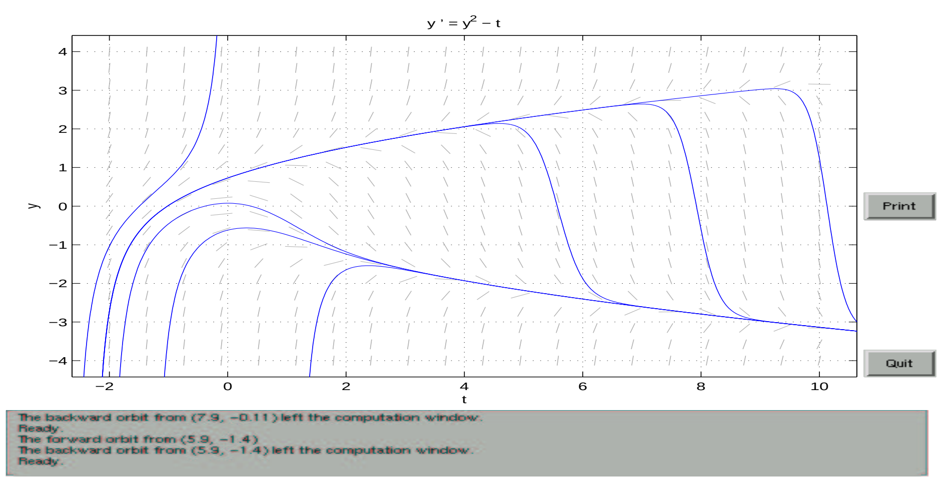
A direction field should be plotted ________________________(1).

Step 1: Draw a region in the $t - y$ plane.

Step 2: Choose a point (a, b) in the region.

Step 3: Plot a short line starting at (a, b) with slope $f(a, b)$.

Step 4: Repeat steps (2) and (3) for many different points (a, b).



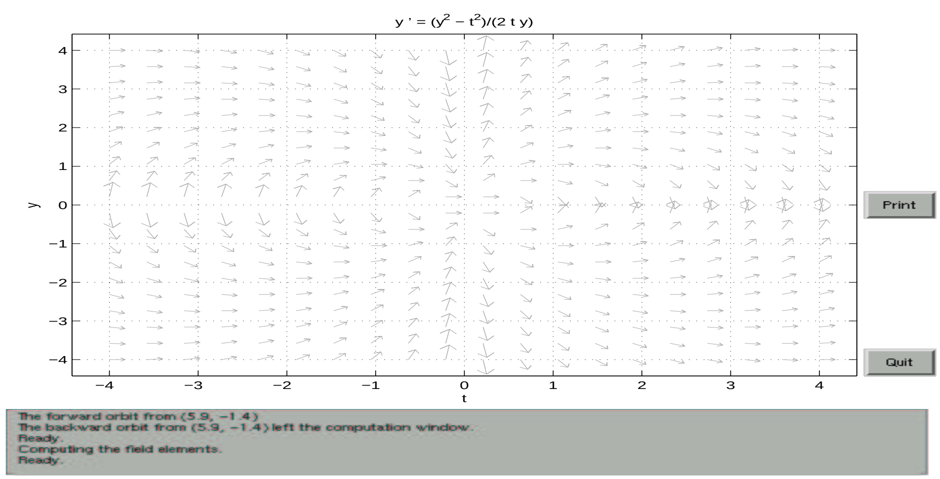
Numerical Solution of ODE IVPs

Direction Field Review – Notes

- A direction field gives a sense of the ____________________(1) of the solutions.
- Warning: be careful plotting a line that has a ____________________(2) (dividing by zero).

  Example:

$$y' = (y^2 - t^2)/(2ty)$$



Numerical Solution of ODE IVPs

Existence and Uniqueness: A Reminder

Questions we must ask:

- Is there a solution?
- Is there only one solution?

Why are these questions important?

1. If there is $\underline{\hspace{4cm}}_{(1)}$, your computer program may $\underline{\hspace{8cm}}_{(2)}$.
2. If there is $\underline{\hspace{8cm}}_{(3)}$, the program will $\underline{\hspace{8cm}}_{(4)}$. It may not be the one you want.

From your ODEs course, you learned theorems that gave checklists ensuring the existence and uniqueness of ODE IVP solutions.

A TIP: Use these theorems.

Numerical Solution of ODE IVPs

Well-Posed Problems: A Must for Computation

Before you begin computing a solution to an IVP, you must ensure that

- the problem is ________________________________ (1)

This automatically ensures that the solution to the problem

- exists
- is unique

Basic meaning: The solution of a *well-posed problem* is not only unique, but also is

________________________________ (2) to

________________________________ (3)

in the data (which almost always will occur on a computer).

Numerical Solution of ODE IVPs

Well-Posed Problems: Formal Definition

Definition: The IVP

$$y' = f(t, y), \;\; a \le t \le b, \;\; y(a) = y_0$$

is a *well-posed* problem if

(1) A ________________ $_{(1)}$ solution $y(t)$ to the problem ________________ $_{(2)}$.

(2) A number $\epsilon > 0$ exists such that a ________________ $_{(3)}$ solution $z(t)$ to the ________________ $_{(4)}$

$$z' = f(t, z) + \delta(t), \;\; a \le t \le b, \;\; z(a) = y_0 + \epsilon_0$$

exists whenever ________________ $_{(5)}$ and ________________ $_{(6)}$

(3) A constant $\kappa > 0$ exists with the property that

$$|z(t) - y(t)| < \kappa\epsilon \;\; \text{for all} \;\; t \in [a, b]$$

Numerical Solution of ODE IVPs

Well-Posed Problems: An Easy Test?

Question: How can we tell whether criteria $\mathbf{(1)-(3)}$ for well-posedness are satisfied for a particular problem?

Answer: Good news! There is an *easy test* (*i.e.,* a theorem) that tells us immediately whether an IVP is well-posed.

Numerical Solution of ODE IVPs

Well-Posed Problems: An Easy Test!

Theorem: Suppose $D = \{(t, y) | t \in [a, b]$ and $y \in [c, d]\}$. The IVP

$$\frac{dy}{dt} = f(t, y), \;\; t \in [a, b], \;\; y(a) = y_0$$

is well-posed provided

1. f is ________________$_{(1)}$ on D
2. f satisfies a ______________________________$_{(2)}$ in the variable ________________$_{(3)}$ on the set ________________$_{(4)}$

Numerical Solution of ODE IVPs

Lipschitz Continuity: Definition

Definition: A function $f(t, y)$ is said to be *Lipschitz continuous* or to satisfy a *Lipschitz condition* in the variable y on a set $D \subset \Re^2$ provided a constant $L > 0$ exists with the property that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

whenever $(t, y_1), (t, y_2) \in D$. The constant L is called the *Lipschitz constant*.

Note: If $f(t, y)$ is differentiable, then the Lipschitz condition guarantees that ____________________$_{(1)}$. Conversely, if f is differentiable with respect to y and ____________________$_{(2)}$, then f satisfies the Lipschitz condition. This property can be used as a ____________________$_{(3)}$ of whether the ____________________$_{(4)}$ is satisfied.

Numerical Solution of ODE IVPs

Lipschitz Condition Test: Example

Example: Determine whether the IVP $y' = f(t,y), \;\; y(t_0) = y_0$ on D is well posed, given

$$f(t,y) = ty$$

and

$$D = \{(t,y) | t \in [1,2],\; y \in [-3,4]\}$$

If so, find the Lipschitz constant.

Answer: For each $(t, y_1),\; (t, y_2)$ in D, we have

$$|f(t,y_1) - f(t,y_2)| = |ty_1 - ty_2| = t|y_1 - y_2| \leq 2|y_1 - y_2|$$

So, $f(t,y)$ is ________________________________$_{(1)}$ in __________$_{(2)}$ on __________$_{(3)}$, with Lipschitz constant $L =$ __________$_{(4)}$.

Numerical Solution of ODE IVPs

Well-Posed versus Stable Solutions (1)

Question: How much can a perturbed solution $\hat{\vec{\mathbf{y}}}(t)$ with perturbed $\hat{\vec{\mathbf{f}}}(t, \hat{\vec{\mathbf{y}}})$ and perturbed initial conditions $\hat{\vec{\mathbf{y}}}(t_0) = \hat{\vec{\mathbf{y}}}_0$ ________________$_{(1)}$ from the original solution $\vec{\mathbf{y}}(t)$ with original $\vec{\mathbf{f}}(t, \vec{\mathbf{y}})$ and original initial conditions $\vec{\mathbf{y}}(t_0) = \vec{\mathbf{y}}_0$ when ________________$_{(2)}$ continuity with constant L on a bounded domain D is assumed?

Answer:

$$||\hat{\vec{\mathbf{y}}}(t) - \vec{\mathbf{y}}(t)|| \leq e^{L(t-t_0)}||\hat{\vec{\mathbf{y}}}_0 - \vec{\mathbf{y}}_0|| + \frac{e^{L(t-t_0)} - 1}{L}||\hat{\vec{\mathbf{f}}} - \vec{\mathbf{f}}||$$

where $||\hat{\vec{\mathbf{f}}} - \vec{\mathbf{f}}|| = \max_{(t,\vec{\mathbf{y}})\in D} ||\hat{\vec{\mathbf{f}}}(t, \vec{\mathbf{y}}) - \vec{\mathbf{f}}(t, \vec{\mathbf{y}})||$.

Numerical Solution of ODE IVPs

Well-Posed versus Stable Solutions (2)

Definition: A solution of the ODE $\vec{\mathbf{y}}' = \vec{\mathbf{f}}(t, \vec{\mathbf{y}})$ is stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\hat{\vec{\mathbf{y}}}(t)$ satisfies the ODE and $||\hat{\vec{\mathbf{y}}}(t_0) - \vec{\mathbf{y}}(t_0)|| \leq \delta$ then $||\hat{\vec{\mathbf{y}}}(t) - \vec{\mathbf{y}}(t)|| \leq \epsilon$ for all $t \geq t_0$.

Example: The solution of the IVP $y' = \lambda y$ with $y(t_0) = y_0$ and λ a constant, is given by $y(t) = y_0 e^{\lambda t}$. So,

- If $\lambda > 0$, every solution is ____________________(1).
- If $\lambda < 0$, every solution is ____________________(2).

Numerical Solution of ODE IVPs

Fundamental Concepts: Euler's Method (1)

Disclaimer: Never use ____________ $_{(1)}$ actually to ____________ $_{(2)}$ an IVP. It is introduced here only to ____________ $_{(3)}$ basic concepts and definitions.

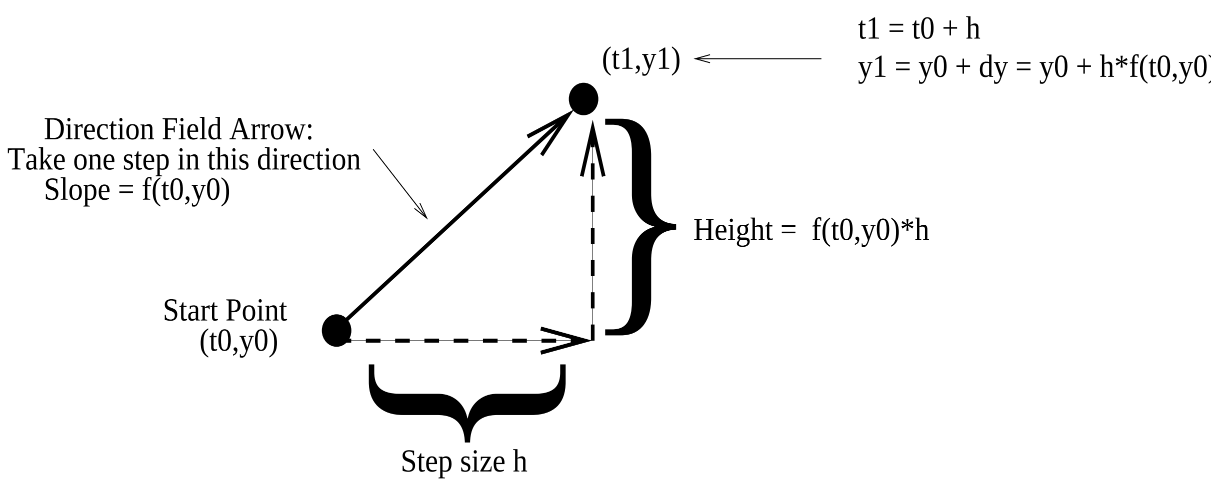
Our canonical IVP:

$$\frac{dy}{dt} = f(t, y), \;\; y(t_0) = y_0$$

on some region D in the plane.

<u>Assumption:</u> All problems we will see are well-posed.

Fundamental Concepts: Euler's Method (2)



In general: Starting at (t_0, y_0), we get to (t_{n+1}, y_{n+1}) by using

$$\begin{aligned} t_{n+1} &= t_n + h \\ y_{n+1} &= y_n + hf(t_n, y_n) \end{aligned}$$

Note: y_{n+1} represents the *numerical approximation* to $y(t_{n+1})$.

Euler's method is: ________________(1) and ________________(2).

Numerical Solution of ODE IVPs

Truncation Error

Truncation Error (TE) arises from the ________________________________(1) of the true solution.

Usually, a ________________(2) or ________________(3) summation approximates an ________________________________(4).

Example: Taylor expand $y(t_{n+1})$ to get Euler's Method (a type of "Taylor Method"):

$$\underbrace{y(t_{n+1}) = y(t_n) + hy'(t_n)}_{\text{Euler's Method: Truncated Series}} \quad + \quad \underbrace{\frac{h^2}{2}y''(t_n) + \mathcal{O}(h^3)}_{\text{Local Truncation Error (LTE)}}$$

Numerical Solution of ODE IVPs

Order of a Method

Generally, Local Truncation Error (LTE) can be approximated by $\alpha h^{\mathbf{k}}$ in the sense that

$$\lim_{h \to 0} \left(\frac{\text{LTE}(h)}{h^{\mathbf{k}}} \right) = \alpha$$

In general: The number $k - 1$ is the order of the numerical method.

Example: Let y_1 be the numerical approximation to $y(t_1)$. Then LTE in Euler's method is:

$$y(t_1) - y_1 = \frac{h^{\mathbf{2}}}{2} y''(t_0) + \cdots = \alpha h^{\mathbf{2}}$$

Here, $k =$ _________ (1), so Euler's method is *order* _________ (2).

Terminology:

An *order* _________ (3) method implies:

The __________________ (4) **error behaves like** _________ (5).

Numerical Solution of ODE IVPs

Summary: Types of Numerical Error

- Rounding error: From finite precision floating point arithmetic. (Example: $\frac{1}{3} \approx 0.3333$)
- Truncation error: From the method used. Two classes:
  - **Local (LTE)**: Error made in $\underline{\hspace{12em}}_{(1)}$ of the numerical method.
  - **Global (GTE)**: Accumulated error. Error made after $\underline{\hspace{24em}}_{(2)}$ of the numerical method.
- A numerical method is "order n" if LTE = $\mathcal{O}(h^{n+1})$.

Numerical Solution of ODE IVPs

Stability of a Numerical Method: Introductory Example

Idea: If ________________________________$_{(1)}$ do not cause the ________________$_{(2)}$ solution to diverge from the ________________$_{(3)}$ solution, the numerical method is stable.

Example: Given the test IVP

$$y' = \lambda y, \;\; y(0) = y_0$$

apply Euler's method with step size ________$_{(4)}$:

$$y_{n+1} = y_n + h\lambda y_n = (1 + h\lambda) y_n$$

which implies that

$$y_{n+1} = \underbrace{(1 + h\lambda)}_{\text{Amplification Factor}}{}^{n+1} y_0$$

Numerical Solution of ODE IVPs

Stability of a Numerical Method: Introductory Example (2)

Example (continued): The "Growth" or "Amplification Factor" = $(1 + h\lambda)$.

Therefore,

- If $|1 + h\lambda| \leq 1$, Euler's method is ________________$_{(1)}$.
- If $|1 + h\lambda| > 1$, Euler's method is ________________$_{(2)}$.

Requirements for Euler to be stable for this example:

- If λ **complex**: $h\lambda$ must be inside a ________________$_{(3)}$ disk in the complex plane centered at ________________$_{(4)}$.
- If λ **real**: $h\lambda$ must be in the interval ________________$_{(5)}$.

Note: Choice of step-size h is crucial for stability.

Numerical Solution of ODE IVPs

Stability of a Numerical Method: General System

General System: ________________ (1)

Taylor Expand: $\vec{\mathbf{y}}(t+h) =$ ________________ (2)

Let $t = t_k$ in Taylor expansion:

$$\vec{\mathbf{y}}(t_{k+1}) = \vec{\mathbf{y}}(t_k) + h\vec{\mathbf{f}}(t, \vec{\mathbf{y}}) + \mathcal{O}(h^2) \tag{1}$$

Euler's method:

$$\vec{\mathbf{y}}_{k+1} = \vec{\mathbf{y}}_k + h\vec{\mathbf{f}}(t, \vec{\mathbf{y}}) \tag{2}$$

Subtract equation (1) from equation (2):

$$\underbrace{\vec{\mathbf{y}}_{k+1} - \vec{\mathbf{y}}(t_{k+1})}_{\text{Global error } e_{k+1}} = (\vec{\mathbf{y}}_k - \vec{\mathbf{y}}(t_k)) + h \underbrace{\left(\vec{\mathbf{f}}(t_k, \vec{\mathbf{y}}_k) - \vec{\mathbf{f}}(t_k, \vec{\mathbf{y}}(t_k))\right)}_{\text{Apply Mean Value Theorem}} - \mathcal{O}(h^2)$$

Numerical Solution of ODE IVPs

Stability of a Numerical Method: General System (2)

Applying the Mean Value Theorem:

$$\vec{\mathbf{f}}(t_k, \vec{\mathbf{y}}_k) - \vec{\mathbf{f}}(t_k, \vec{\mathbf{y}}(t_k)) = \mathbf{J}_f(t_k, \alpha\vec{\mathbf{y}}_k + (1-\alpha)\vec{\mathbf{y}}(t_k))(\vec{\mathbf{y}}_k - \vec{\mathbf{y}}(t_k))$$

where

- $\mathbf{J}_f =$ ____________________$_{(1)}$ matrix of $\vec{\mathbf{f}}$ w.r.t. $\vec{\mathbf{y}}$ and $\alpha \in [0, 1]$.

____$_{(2)}$ is expressed in general as:

$$\vec{\mathbf{e}}_{k+1} = \underbrace{(\mathbf{I} + h\mathbf{J}_f)}_{\text{Amplification Factor}} \vec{\mathbf{e}}_k + \text{LTE}_{k+1}$$

Requirement for ____________________$_{(3)}$:

$$\rho(\mathbf{I} + h\mathbf{J}) \leq 1$$

where ρ represents the ____$_{(4)}$ of a matrix.

Numerical Solution of ODE IVPs

Stability of a Numerical Method: General System (3)

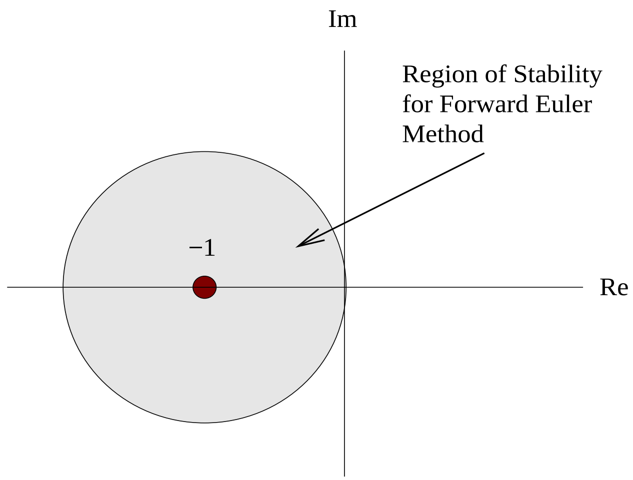
- Observation: Stability requires that $\rho(\mathbf{I} + h\mathbf{J}) \leq 1$.
- Question: What does this stability restriction imply?
- Answer: All eigenvalues of $h\mathbf{J}_f$ must lie inside a $\underline{\hspace{4cm}}_{(1)}$ disk in the $\underline{\hspace{8cm}}_{(2)}$ centered at $\underline{\hspace{2cm}}_{(3)}$.
- Note: If eigenvalues lie $\underline{\hspace{4cm}}_{(4)}$ the disk, the method will be $\underline{\hspace{4cm}}_{(5)}$.
- Implication: We must choose $\underline{\hspace{8cm}}_{(6)}$ so that all stability constraints are satisfied.

Numerical Solution of ODE IVPs

Stability of a Numerical Method:

Euler Method Region of Stability

All eigenvalues of $h\mathbf{J}_f$ must lie inside the disk.



Stability of a Numerical Method: Euler Example

Example: Consider

$$y' = -10(t-1)y, \;\; y(0) = e^{-5}, \;\; t \in [0,2]$$

Question: When will Euler's Method be stable?

Answer: Notice that EM implies

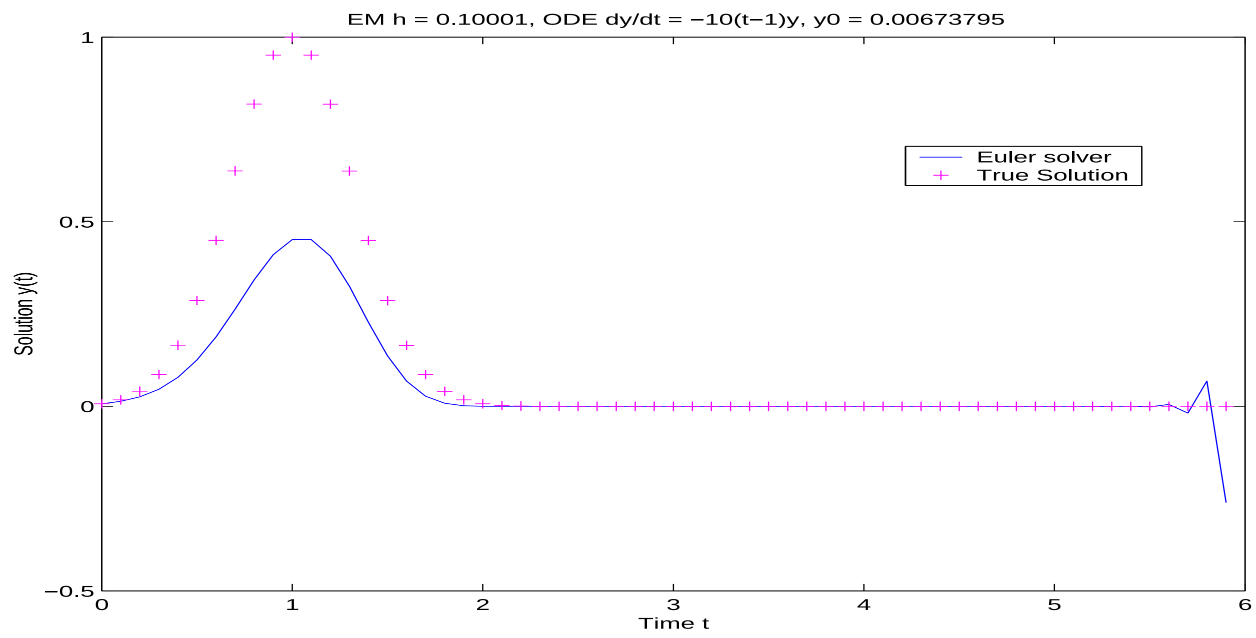
$$y_{k+1} = y_k + h(-10(t-1)y_k) = \underbrace{(1 - 10h(t-1))}_{\text{Amplification Factor}} y_k$$

Therefore,

- For $\_\_\_\_\_\_\_\__{(1)}$, the method is unstable for any $\_\_\_\_\_\_\_\__{(2)}$.
- For $\_\_\_\_\_\_\_\__{(3)}$, the method will be stable if $\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\__{(4)}$.

Numerical Solution of ODE IVPs

Stability of a Numerical Method: Euler Example Illustration



Numerical Solution of ODE IVPs

Implicit Methods

- Recall: Euler's method (EM) is ____________________$_{(1)}$ and ____________________$_{(2)}$. The limited region of stability for EM requires we choose ____________________$_{(3)}$ carefully.
- Improvement: Make the ____$_{(4)}$ ____________________$_{(5)}$.
- How? Use information at __________$_{(6)}$ as wells as at __________$_{(7)}$.
- This makes the method ____________________$_{(8)}$.

Numerical Solution of ODE IVPs

Implicit Methods: Backward Euler

- Example of an Implicit Method: *Backward Euler Method (BE)*

$$\vec{\mathbf{y}}_{k+1} = \vec{\mathbf{y}}_k + h\vec{\mathbf{f}}\underbrace{(t_{k+1}, \vec{\mathbf{y}}_{k+1})}_{\text{Implicit}}$$

- Question: How can we solve for $\vec{\mathbf{y}}_{k+1}$?

- Answer:

    - Use __________________$_{(1)}$. This often requires calculating the ____________________________________$_{(2)}$ of the function $\vec{\mathbf{f}}(t, \vec{\mathbf{y}})$.

    - Use __________________$_{(3)}$ methods.

- Note: Both approaches require an initial guess, usually derived by taking one step of an explicit method.

Numerical Solution of ODE IVPs

Implicit Methods: Stability

- Trade-off: Implicit methods take $\underline{\hspace{6cm}}_{(1)}$ but are more $\underline{\hspace{6cm}}_{(2)}$.
- Apply BE to the test ODE $y' = \lambda y$:

$$\begin{aligned} y_{k+1} &= y_k + h\lambda y_{k+1} \\ \text{Therefore} \Rightarrow y_k &= \underbrace{\left(\frac{1}{1-h\lambda}\right)^k}_{\text{Amplification Factor}} \; y_0 \end{aligned}$$

- For BE to be stable we require

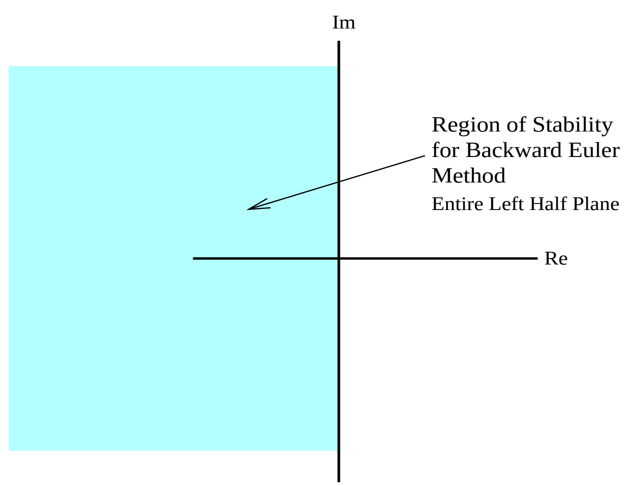
$$\left|\frac{1}{1-h\lambda}\right| < 1$$

- Good news! As long as $\Re(\lambda) < 0$, BE is stable for any step size h.

Numerical Solution of ODE IVPs

Stability of a Numerical Method: BE Stability Region Illustration

All eigenvalues of $h\mathbf{J}_f$ must lie in the left half plane.



Numerical Solution of ODE IVPs

Implicit Methods: Stability Notes

Notes:

- BE is $\underline{\hspace{4cm}}_{(1)}$ accurate (just like $\underline{\hspace{4cm}}_{(2)}$)
- For a $\underline{\hspace{4cm}}_{(3)}$, the stability requirement becomes $\rho((\mathbf{I} - h\mathbf{J}_f)^{-1}) < 1$
- The stability region for BE is the entire $\underline{\hspace{4cm}}_{(4)}$ of the complex plane (as compared to the radius 1 disk of EM).
- Since $\underline{\hspace{2cm}}_{(5)}$ step size h keeps us within the stability region, the BE method is called $\underline{\hspace{8cm}}_{(6)}$.
- There exist higher order implicit methods, like the Trapezoid Method (order = $\underline{\hspace{2cm}}_{(7)}$).
- Not all implicit methods are $\underline{\hspace{8cm}}_{(8)}$.

Numerical Solution of ODE IVPs

Stiffness in ODEs

Question: What is "stiffness"?

Answer:

- Physically: A process whose components have __________________$_{(1)}$ time scales. Also, a process whose time scale is __________________$_{(2)}$ compared to the time interval over which it is being observed.

- Mathematically: A well-posed ODE $\vec{\mathbf{y}}' = \vec{\mathbf{f}}(t, \vec{\mathbf{y}})$ is "stiff" if its Jacobian $\mathbf{J}_f$ has __________________$_{(3)}$ that differ greatly in magnitude.

- Practically: An ODE is stiff if an explicit method (like EM) is __________________$_{(4)}$ because stability requirements force the step size h to be extremely small.

Stiffness in ODEs: Example 1

Example 1: For test ODE $y' = \lambda y$ with $t \in [a, b]$, the problem is

$\underline{\qquad\qquad\qquad\qquad}_{(1)}$ if $\underline{\qquad\qquad\qquad\qquad}_{(2)}$.

Recall: For stability in EM, we require $|1 - h\lambda| < 1$. If λ is real and $\lambda << -1$, this forces $\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{(3)}$.

For a **system**: A system of ODEs with $t \in [a, b]$ is $\underline{\qquad\qquad\qquad\qquad}_{(4)}$ when

$\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{(5)}$ where λ_j are the

$\underline{\qquad\qquad\qquad\qquad}_{(6)}$ of Jacobian $\mathbf{J}(t, \vec{\mathbf{y}}(t))$.

Numerical Solution of ODE IVPs

Stiffness in ODEs: Example 2

Example 2: Consider

$$y' = -\alpha(y - \sin t) + \cos t, \;\; y(0) = 1, \;\; t \in [0, 1]$$

Let $\alpha = 1000$. Then the Jacobian is _________________$_{(1)}$. So eigenvalue $\lambda =$ _________________$_{(2)}$. Therefore,

$$(1 - 0)\Re(\lambda) = (-\alpha) = -1000 << -1$$

This is _________________$_{(3)}$ on $t \in [0, 1]$.

Note: on another interval, $t \in [0, 0.002]$ we have

$$(0.002 - 0)\Re(\lambda) = (0.002)(-1000) = -2$$

so this ODE is _________________$_{(4)}$ on this interval.

Numerical Solution of ODE IVPs

Stiffness: EM vs BE

Consider

$$y' = -100y + 100t + 101, \;\; y(0) = 1$$

Let $h = 0.1$. Computed output with perturbed initial data:

Time t	0.00	0.10	0.20	0.30	0.40
Exact Soln	$\mathbf{1.00}$	$\mathbf{1.10}$	$\mathbf{1.20}$	$\mathbf{1.30}$	$\mathbf{1.40}$
EM	0.99	1.19	0.39	8.59	-64.21
EM	1.01	1.01	2.01	-5.99	67.01
BE	0.00	1.01	1.19	1.30	1.40
BE	2.00	1.19	1.21	1.30	1.40

Numerical Solution of ODE IVPs

Stiffness: Comments on EM vs BE

Note:

- EM ____________________$_{(1)}$ with only ____________________$_{(2)}$ perturbations.
- BE is ____________________$_{(3)}$ even with ____________________$_{(4)}$ perturbations.

Numerical Solution of ODE IVPs

Stiffness: Summary

- A particular ODE may be ________________$_{(1)}$ or ________________$_{(2)}$.
- A numerical ODE solving method can be ________________$_{(3)}$ or ________________$_{(4)}$ for a particular problem.
- The stability of the numerical method often depends on ________________________________$_{(5)}$.
- ________________$_{(6)}$ methods should (almost) always be used to solve stiff ODEs.

Other IVP Solvers

Other classes of numerical IVP solvers include (but are not limited to):

- Higher Order Taylor methods (seldom used)
  - Can give ____________________$_{(1)}$ accuracy.
  - They require the computation of the ____________________$_{(2)}$ of $f(t,y)$.
- Runge-Kutta methods (very popular)
  - Can give ____________________$_{(3)}$ accuracy.
  - Do not need ____________________$_{(4)}$ of $f(t,y)$.
  - Can be ____________________$_{(5)}$ or ____________________$_{(6)}$.
  - These are ____________________$_{(7)}$-step methods.
  - Methods include: Midpoint, Modified Euler, Heun, 4th Order Runge-Kutta (RK4)

Numerical Solution of ODE IVPs

Other IVP Solvers (cont)

- Multi-step methods
  - Can give ________________$_{(1)}$ accuracy.
  - Can be ________________$_{(2)}$ or ________________$_{(3)}$.
  - Starting values must be calculated with a ________________$_{(4)}$ method (*e.g.*, RK4)
  - Methods include: Adams-Bashforth, Adams-Moulton, Milne, Simpson
- Extrapolation methods
  - These take solutions generated by lower order methods, and increase ________________$_{(5)}$ by ________________$_{(6)}$.
  - Variations of these methods presented in [SB93], [Ste73], and [Gra65]. Also see the discussion and reference list in [Asa95, p.642].

Numerical Solution of ODE IVPs

Systems and Higher Order IVPs

- All methods and theories presented can be ____________________(1) to apply to systems.
- Many higher order IVPs can be ____________________(2) to 1st order systems of IVPs. Then all methods and theories apply here, too.

Example: Suppose we have the second order equation describing a linear spring,

$$y'' = -ky, \quad y(0) = y_0, \quad y'(0) = v_0$$

Convert this to a 2×2 first order system of equations:

$$\begin{aligned} y' &= v \\ v' &= \underline{\hspace{5cm}}_{(3)} \end{aligned}$$

Numerical Solution of ODE IVPs

Solving IVPs with Packaged Software

To solve a system of ODE IVPs $\vec{\mathbf{y}}' = \vec{\mathbf{f}}(t, \vec{\mathbf{y}})$ with $y(t_0) = y_0$ with a packaged routine typically requires ________________$_{(1)}$ to supply the following:

- The name of the routine that computes $\vec{\mathbf{f}}(t, \vec{\mathbf{y}})$.
- ________________$_{(2)}$ and ________________$_{(3)}$ values for times t.
- Initial value y_0.

$\cdots$ and for some solvers, sometimes $\cdots$

- The number of equations in the system.
- ________________$_{(4)}$ and/or ________________$_{(5)}$ error tolerances.

$\cdots$ and sometimes for a stiff ODE $\cdots$

- The routine that computes the Jacobian $\mathbf{J}_f$ of function $\vec{\mathbf{f}}$.