Solutions to Assignment #2

1. Consider the population model given by the difference equation

$$N_{t+1} - N_t = m,$$

where m is a constant, for $t = 0, 1, 2, \ldots$.

   (a) Give an interpretation for this model.

   Solution: This equation says that the population increment (or decrease, if $m < 0$) is constant; equivalently, after each unit of time, the same number of individuals are added (or taken away from) to the population. □

   (b) If the initial population density is N_o, what does this model predict in the long run? Consider the two possibilities $m < 0$ and $m > 0$.

   Solution: From $N_{t+1} = N_t + m$ we get tht $N_1 = N_o + m$. Consequently, $N_2 = N_1 + m = N_o + m + m = N_o + 2m$. Similarly, $N_3 = N_o + 3m$. Thus, it follows by induction on n that $N_n = N_o + nm$ for all $n = 1, 2, 3, \ldots$ Hence

$$N_t = N_o + mt \quad \text{for all} \ \ t = 1, 2, 3 \ldots$$

   Hence, if $m > 0$, the population will increase linearly and indefinitely, while if $m < 0$, it will decrease to extinction in a finite time. □

   (c) How does this model compare with the Malthusian model?

   Solution: This model predits linear growth or decay, while the Malthusian model predits geometric growth or decay. □

2. Assume that the *per–capita* growth rate λ of a population is less than 1; that is, left on its own, the population will go extinct. To avoid extinction, suppose that after each unit of time, a constant number m of individuals of the same species is added to the population.

   (a) Write down a difference equation that models this situation.

   Solution: $N_{t+1} = \lambda N_t + m$.

   (b) Solve the difference equation and discuss what this model predicts in the long run.

   Solution: Suppose that at time $t = 0$ there are N_o individuals. Then, $N_1 = \lambda N_o + m$. Thus, $N_2 = \lambda N_1 + m = \lambda(\lambda N_o + m) + m = \lambda^2 N_o + \lambda m + m$.

In a similar manner we can compute $N_3 = \lambda^3 N_o + \lambda^2 m + \lambda m + m$. Hence, by induction on n we can show that

$$\begin{aligned} N_n &= \lambda^n N_o + \lambda^{n-1} m + \lambda^{n-2} m + \cdots + \lambda m + m \\ &= N_o \lambda^n + m\left(\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda + 1\right) \\ &= N_o \lambda^n + m \cdot \tfrac{\lambda^n - 1}{\lambda - 1} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Consequently,

$$N_t = N_o \lambda^t + m \cdot \frac{1 - \lambda^t}{1 - \lambda} \quad \text{for} \quad t = 0, 1, 2, \ldots$$

Now, since $|\lambda| < 1$ it follows that

$$\lim_{t \to \infty} N_t = \frac{m}{1 - \lambda}.$$

Thus, this model predicts that the population will tend to the equilibrium value of $m/(1 - \lambda)$ □

(c) How does this model compare with the Malthusian model?

Solution: While the Malthusian model (with $\lambda < 1$) predicts extinction, this model predicts that the population will tend towards a non-zero steady state. □

3. *[Problem 1.1.2 on page 6 in Allman and Rhodes].* In early stages of the development of a frog embryo, cell division at a fairly regular rate. Suppose that you observe that all cells divide, and hence the number of cells doubles, roughly every half hour.

(a) Write down an equation modeling this situation.

Solution: Let N_t denote the number of cells in the embryo at time t, where t denotes the number of doubling times; that is, t is measured in mumbers of 30–minute periods. Assume also that there is one cell ($N_o = 1$) at the start of the process. Then, the difference equation modeling the growth of the embryo is

$$N_{t+1} = 2N_t. \quad \square$$

(b) Produce a table and graph the number of cells in the embryo as a function of t.

Solution: Figure 2 shows the graph. □

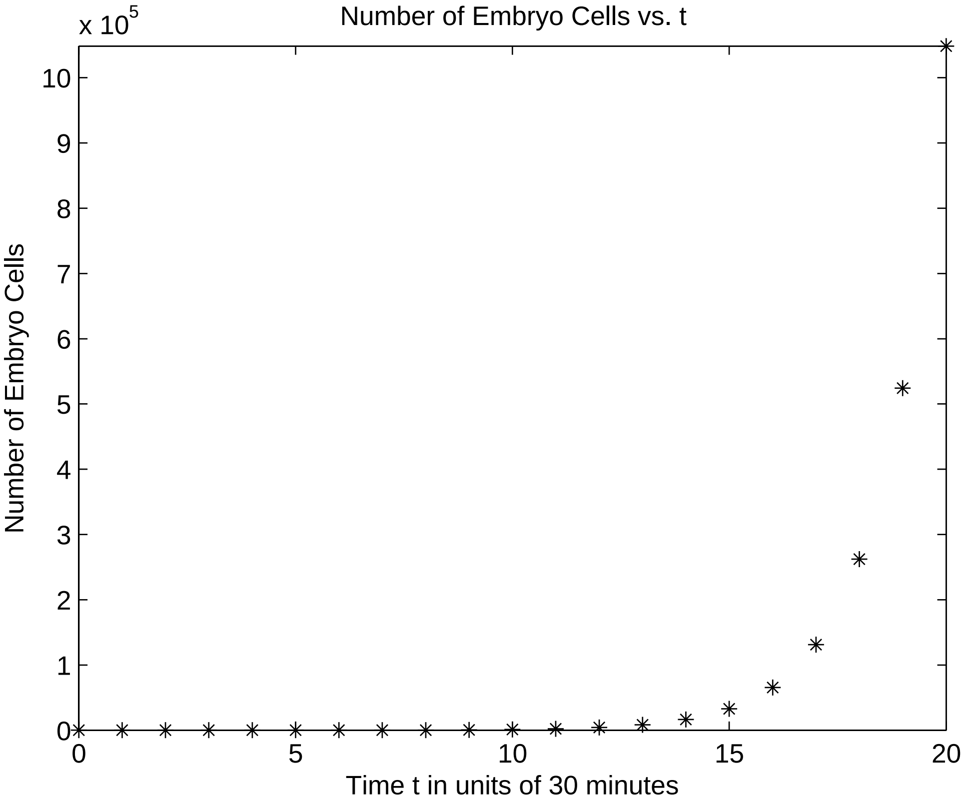

Figure 1: Graph for Problem 1.1.2 part (b)

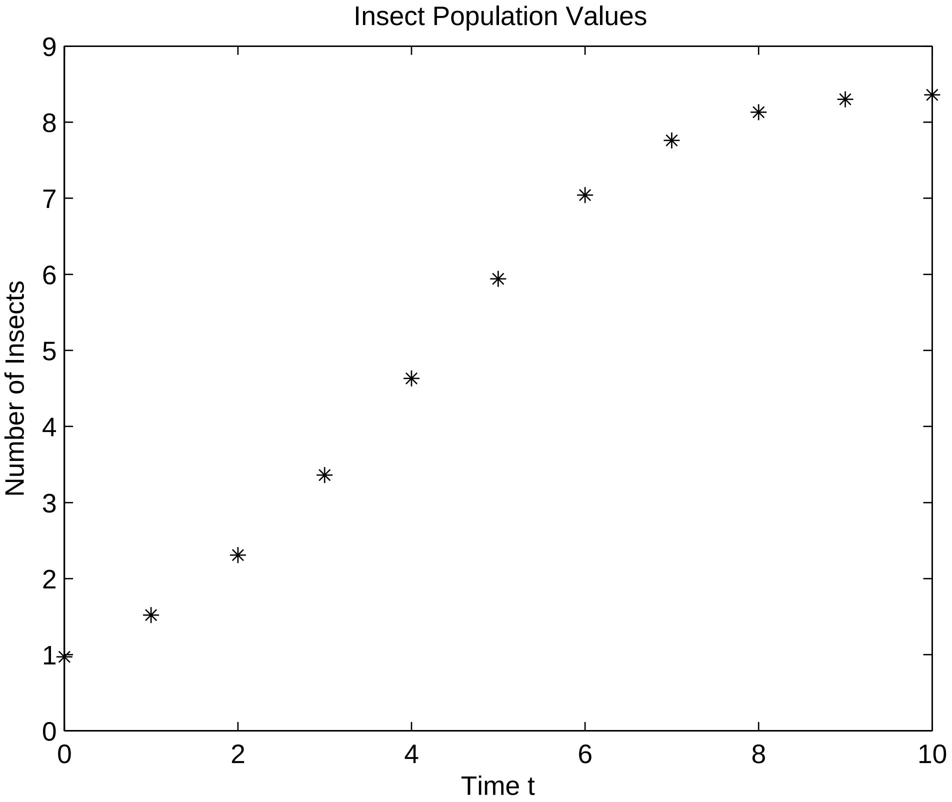

Figure 2: Plot of Insect Population Values on p. 7 of Allman and Rhodes

(c) Further observation shows that, after 10 hours, the embryo has 30,000 cells. Is this roughly consistent with the model? What biological conclusions and/or questions does this raise?

Solution: A time period of 10 hours corresponds to $t = 20$. The predicted value then is $N_10 = 2^{20} = 1,048,576$. There is therefore a large discrepancy suggesting that a simple geometric growth model is not the appropriate one for embryo cells. Perhaps, after several divisions, cells especialize and differentiate and therefore might take longer to divide. □

4. *[Problem 1.1.6 on page 7 in Allman and Rhodes].*

Solution: Figure 3 shows the graph of the insects population values versus t in Table 1.2 on p. 7 of Allman and Rhodes. Insect growth is definitely not consistent with the geometric growth model. Perhaps, this might be the case over the time interval $[0, 4]$. However, the logistic model seems to be more appropriate in this case. □

5. *[Problem 1.1.10 on page 7 in Allman and Rhodes].* A model for the growth of P_t is said to have a *steady state* or *equilibrium point* at P^* if whenever $P_t = P^*$, then $P_{t+1} = P^*$.

   (a) This is equivalent to saying that: P^* is a steady state if, whenever $P_t = P^*$, then $\Delta P = 0$. □

   (b) More intuitively, P^* is a steady state, if whenever the value P^* is reached, the population values remain at P^* for all values of t. □

   (c) Can a model described by $P_{t+1} = (1 + r)P_t$ have a steady state? Explain. *Solution:* Suppose there is a steady state P^*. It then follows that $P^* = (1 + r)P^*$, which implies that $1 = 1 + r$, and therefore $r = 0$. Thus, there is a steady state only when $r = 0$. Notice that in this case we get the difference equation $P_{t+1} = P_t$ which can only have constant solutions. □