

Solutions to Assignment #3

1. [Problem 1.1.11 on page 8 in Allman and Rhodes] Explain why the model

$$\Delta P = rP$$

leads to the formula

$$P_t = (1 + r)^t P_o.$$

Solution: Write the equation in the form

$$P_{t+1} = (1 + r)P_t, \tag{1}$$

and assume that P_t is P_o when $t = 0$. Then, by (1),

$$P_1 = (1 + r)P_o,$$

and therefore, using (1) again (this time for $t = 1$),

$$P_2 = (1 + r)P_1 = (1 + r)(1 + r)P_o = (1 + r)^2 P_o.$$

We may now proceed by induction on n . Assume therefore that we have established that

$$P_n = (1 + r)^n P_o.$$

Then, applying (1) with $t = n$,

$$P_{n+1} = (1 + r)P_n = (1 + r)(1 + r)^n P_o = (1 + r)^{n+1} P_o.$$

Hence, by the Principle of Mathematical Induction,

$$P_t = (1 + r)^t P_o \quad \text{for } t = 0, 1, 2, \dots \quad \square$$

2. [Problem 1.2.7 on page 18 in Allman and Rhodes]

Solution: Figure 1 shows the plot of the insect population data displayed in Table 1.6, page 18, of Allman and Rhodes. It shows a typical S -curve associated with the logistic growth model. By inspecting the graph, we can get an estimate for the carrying capacity of the population, K , of about 8.5 (Note: the horizontal line at $N = 8.5$ is also shown in the graph). To estimate r we may take the first two data points and compute ΔN in the equation

$$\Delta N = rN_t \left(1 - \frac{N_t}{K} \right).$$

Taking $t = 0$, so that $\Delta N = N_1 - N_o = 1.52 - 0.97 = 0.55$, and $K = 8.5$, we can estimate r from the last equation by solving for it

$$r = \frac{\Delta N}{N_o \left(1 - \frac{N_o}{K} \right)} \approx \frac{0.55}{0.97 \left(1 - \frac{0.97}{8.5} \right)} \approx 0.64. \quad \square$$

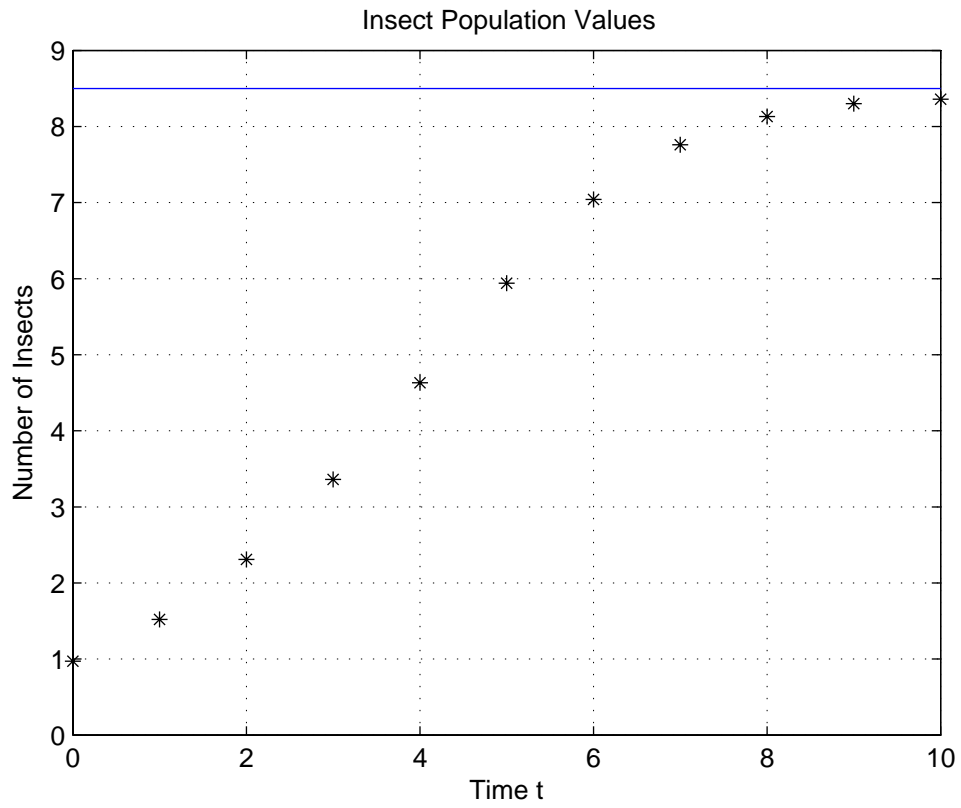


Figure 1: Plot of Insect Population Values in Table 1.6 of Allman and Rhodes, p. 18

3. [Problem 1.2.8 on page 18 in Allman and Rhodes] Suppose the growth of a population is modeled by the equation

$$N_{t+1} = N_t + 0.2N_t \left(1 - \frac{N_t}{200000}\right), \quad (2)$$

where N_t is measured in *individuals*.

- (a) Find an equation of the same form, describing the same model, but with the population measured in *thousands of individuals*.

Solution: Let $M_t = \frac{1}{1000}N_t$ for each $t = 0, 1, 2, \dots$. Divide equation (??) by 1000 and rearrange the last term to get

$$\frac{N_{t+1}}{1000} = \frac{N_t}{1000} + 0.2\frac{N_t}{1000} \left(1 - \frac{N_t/1000}{200}\right).$$

Thus, the equation for M_t reads

$$M_{t+1} = M_t + 0.2M_t \left(1 - \frac{M_t}{200}\right). \quad \square$$

- (b) Find the equation of the same form, describing the same model, but with the population measured in units chosen so that the carrying capacity is 1 in those units

Solution: This time we let $M_t = \frac{1}{200000}N_t$; that is, we divide N_t by the carrying capacity. Then, proceeding as in the previous example (this time dividing by 200000) we obtain

$$M_{t+1} = M_t + 0.2M_t(1 - M_t). \quad \square$$

4. (*US Census Data.*) The MS Excel file CensusDataUS in the Math 36 webpage (see the courses website at <http://pages.pomona.edu/~ajr04747>) contains the total US population (in millions of people) for each year that a census has been taken in the United States.

- (a) Use MATLAB[®] to get a plot of the US population as a function of t , where t is in units of 10 years since the year 1790.

Solution: Figure 3 shows the plot. \square

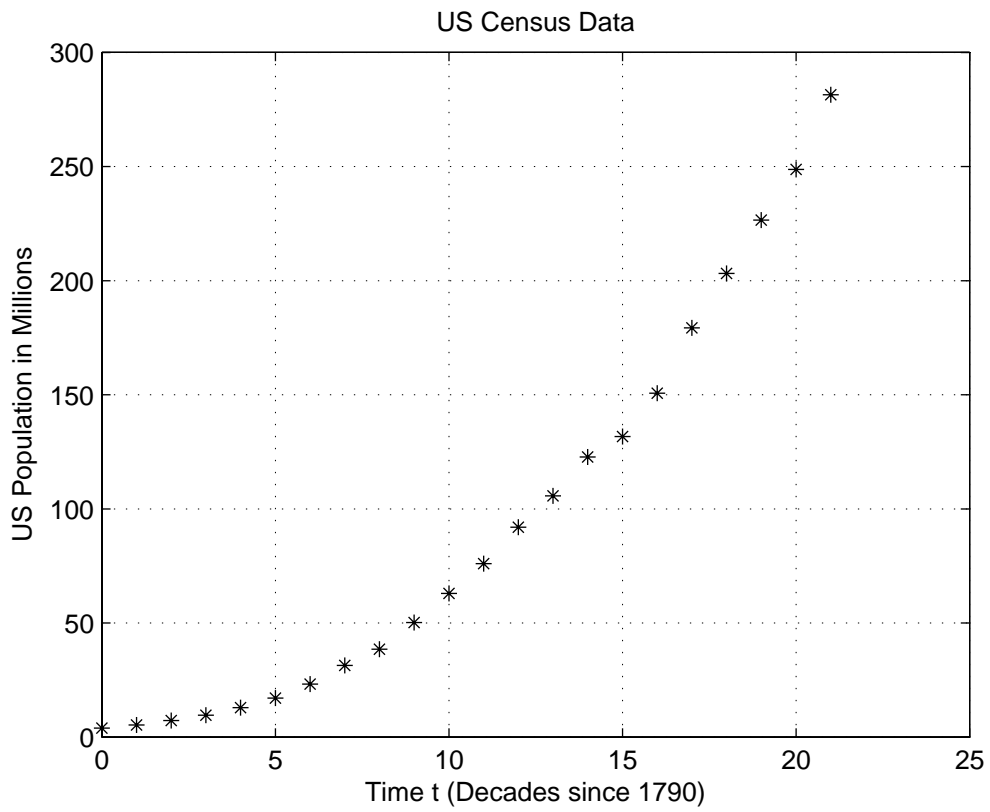


Figure 2: US Census Data since 1790

- (b) If the US population follows a Malthusian model, what would the growth rate λ be? Using this value of λ , compute the population values that the model predicts for $t = 1, 2, 3, \dots$. Plot the predicted and actual values on the same graph. How well do these predictions compare with the actual data?

Solution: If we let N_t denote the US population (in millions) for $t = 0, 1, 2, \dots$, the Malthusian model predicts that $N_t = N_o \lambda^t$ for $t = 0, 1, 2, \dots$. We can take N_o to be the US population in 1790; i.e., $N_o = 3.929$. We can estimate λ by computing

$$\lambda = \frac{N_1}{N_o} \approx \frac{5.308}{3.929} \approx 1.351.$$

We use these values of N_o and λ to compute the values predicted by the Malthusian model. These values are plotted in Figure 4 (solid curve with '+' for the predicted values) along with the actual US Census data. We can see that the predicted values diverge a great deal from the actual census data. \square

5. (*US Census Data, continued*). Starting with the solution to the Malthusian model: $N_t = N_o \lambda^t$, take logarithms on both sides to get

$$\ln N_t = \ln N_o + t \ln(\lambda).$$

Thus, the relationship between $\ln N_t$ and t should be linear with slope $\ln(\lambda)$ and y -intercept $\ln N_o$.

- (a) If \mathbf{X} represents a row of values, and \mathbf{Y} another row of values of the same size, the MATLAB[®] function `polyfit(X, Y, 1)` returns the slope m and y -intercept y_o of the line that best fits the data (in the sense of least squares regression) in \mathbf{X} and \mathbf{Y} :

$$y = mx + y_o.$$

Use this function to obtain estimates for the values of $\ln N_o$ and $\ln(\lambda)$

Solution: Define `t = [0:21]` and `Y = log(USpop)` in MATLAB[®], where `USpop` is the array containing the US Census data. Then, `polyfit(t, Y, 1)` returns the slope `m` and y -intercept `b` for the least-squares regression line. The MATLAB[®] output yields $m = 0.2019$ and $b = 1.8023$. \square

- (b) Obtain estimates for N_o and λ , and use them to generate a new set of predicted values for the US population. Plot these, along with the actual data, and assess how good the fit is.

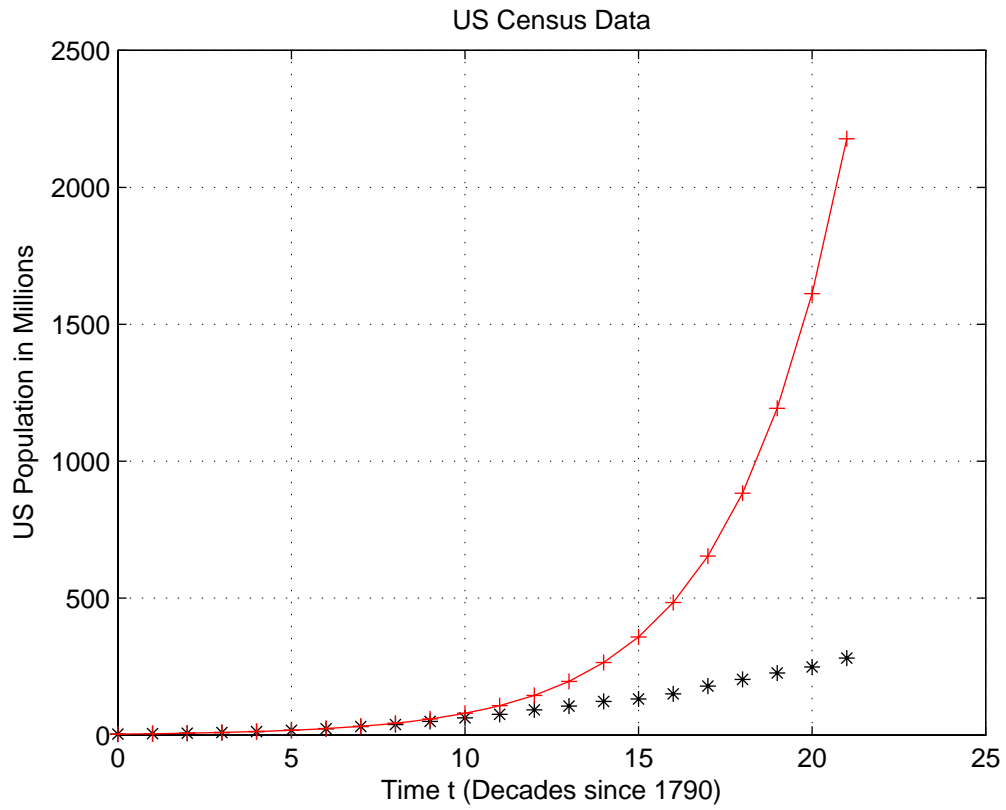


Figure 3: US Census Data and Values Predicted by Malthusian Model

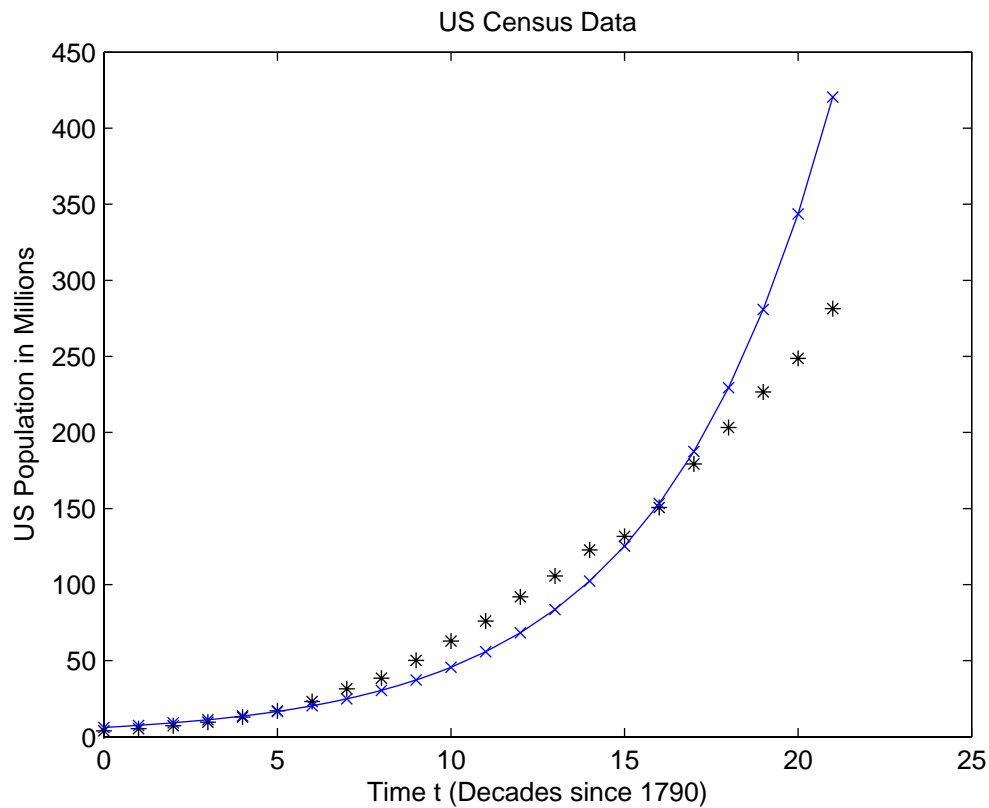


Figure 4: US Census Data and Values Predicted by Malthusian Model

Solution: Using the values for m and b obtained in the previous part, we get the following estimates for λ and N_o : $\lambda = \exp(m) \approx 1.2237$ and $N_o = \exp(b) \approx 6.0634$. As in the previous problem, we can use these values to obtain predicted population values according to the Malthusian model. These values are plotted in Figure 5 (solid curve with 'x' for the predicted values) along with the actual US Census data. We see by inspecting the graph that the fit, though better than the one in the previous problem, is actually not very good.