

Solutions to Assignment #6

1. [Problems 1.2.9 and 1.2.10 on page 18 in Allman and Rhodes]

1.2.9 Graphically determine the populations for the next six time increments in each of the models in Figure 1 using the initial population shown.

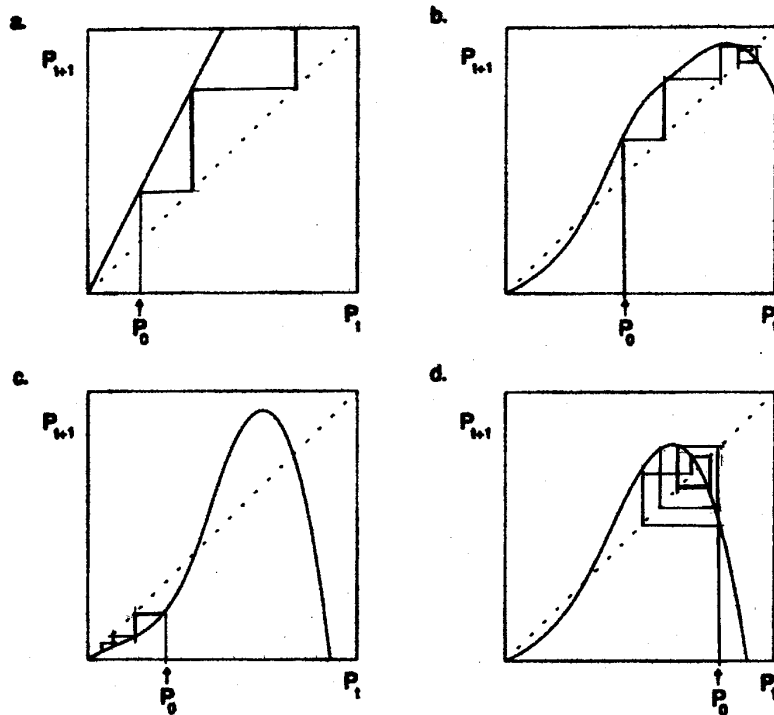


Figure 1: Cobweb graphs for problems 1.2.9 and 1.2.10 in Allman and Rhodes

*Solution:* The solutions are also shown in Figure 1. □

1.2.10 Give a formula for the model appearing in part (a) of Figure 1. What is the name of this population model?

*Solution:* Part (a) of Figure 1 describes the model  $P_{t+1} = f(P_t)$ , where  $f$  is the linear function  $f(P) = \lambda P$ , for some  $\lambda > 1$ . This is the Malthusian model. □

2. [Problems 1.3.1 and 1.3.2 on page 28 in Allman and Rhodes]

1.3.1 The equilibrium points of a model,  $P_{t+1} = f(P_t)$ , are located where the graph of  $P_{t+1} = f(P_t)$  crosses the line  $P_{t+1} = P_t$  in the  $P_t - P_{t+1}$ -plane. Suppose we focus on a section of the graph around an equilibrium point and zoom in so that the graph of  $P_{t+1} = f(P_t)$  appears to be a straight line (i.e., we are looking at the linearization of  $f$  around the equilibrium point). In each of the models of Figure 2, decide whether the equilibrium point is stable or unstable by choosing a  $P_o$  close to the steady state and then cobwebbing.

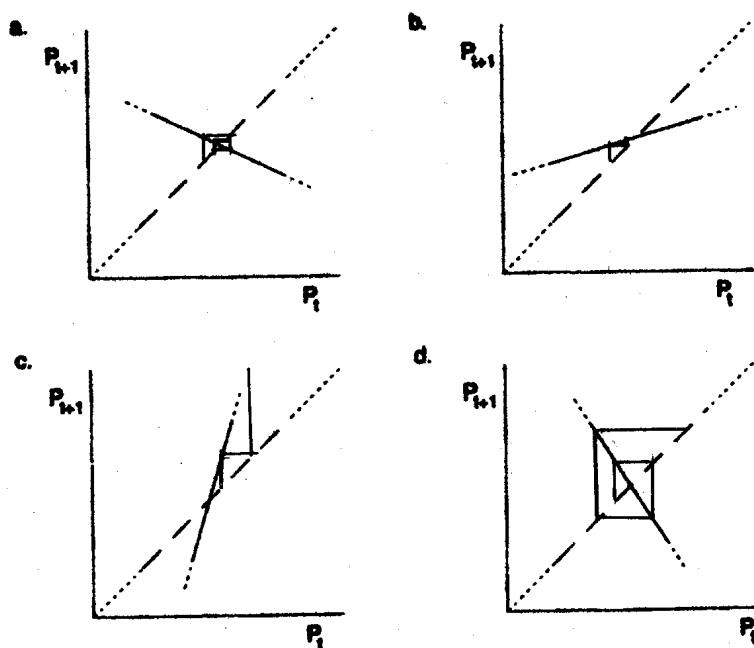


Figure 2: Cobweb graphs for problems 1.3.1 and 1.3.2 in Allman and Rhodes

*Solution:* The cobweb diagrams are also shown in Figure 2. According to the figure, the equilibrium points in parts (a) and (b) are stable, while those in parts (c) and (d) are unstable. □

1.3.2 Reasoning from the previous problem, in what range must the slope of the graph of  $P_{t+1} = f(P_t)$  be at an equilibrium point to produce stability? Instability?

*Solution:* If  $m$  is the slope of the graph of  $P_{t+1} = f(P_t)$  be at an equilibrium point, for stability  $-1 < m < 1$ , and for instability  $m > 1$  or  $m < -1$ .

□

3. (*US Census Data, Revisited.*) In this problem and the next, we fit a logistic curve to the US Census data contained in the MS Excel file `CensusDataUS.xls` in the Math 36 website (<http://pages.pomona.edu/~ajr04747>)

The idea for this fit is to observe that, if we write the logistic difference equation

$$N_{t+1} = N_t + rN_t \left(1 - \frac{N_t}{K}\right)$$

in the form  $\frac{\Delta N}{N_t} = r - \frac{r}{K}N_t$ , where  $\Delta N = N_{t+1} - N_t$ , then the logistic model predicts that the relationship between the *relative increments*  $\frac{\Delta N}{N_t}$  and  $N_t$  should be linear with slope  $-r/K$  and  $y$ -intercept  $r$ . Thus, the parameters  $r$  and  $K$  can be estimated from the data by a linear, least-squares regression fit of the relative increments versus the population density.

- (a) Use MATLAB<sup>®</sup> to define an array,  $\mathbf{Y}$ , made up of the relative increments of the US population since census started being taken. The size of this new array should be one less than the size of the US population array.
- (b) Define an array,  $\mathbf{N}$ , made up of the US population values up to the next to the last one (i.e., the census values from 1790 to 1990).
- (c) Plot  $\mathbf{Y}$  versus  $\mathbf{N}$ . Use the MATLAB<sup>®</sup> command `plot(N,Y,'k*')` and then type `hold on` in the command window in order to keep the plot.

*Solution:* Figure 3 shows the graphs of  $\mathbf{Y}$  versus  $\mathbf{N}$  and the least-squares regression line.

- (d) Use the MATLAB<sup>®</sup> command `polyfit(N,Y,1)` to obtain the slope,  $m$ , and  $y$ -intercept,  $b$ , of the least-squares regression line of  $\mathbf{Y}$  versus  $\mathbf{N}$ , and sketch this line on the same graph obtained in the previous part.

*Solution:* Figure 3 shows the graphs of  $\mathbf{Y}$  versus  $\mathbf{N}$  and the least-squares regression line.

- (e) Use the slope and  $y$ -intercept obtained in the previous part to estimate the intrinsic growth rate,  $r$ , and carrying capacity,  $K$ , for the US population.

4. (*US Census Data, Revisited (continued).*)

- (a) Use the estimates for  $r$  and  $K$  obtained in the previous problem, and the US population in 1790 as  $N_o$ , to compute population values predicted by the logistic model for each of the decades since 1790 until 2000. You may use the MATLAB<sup>®</sup>.m-file `LogisticK.m` to do these calculations

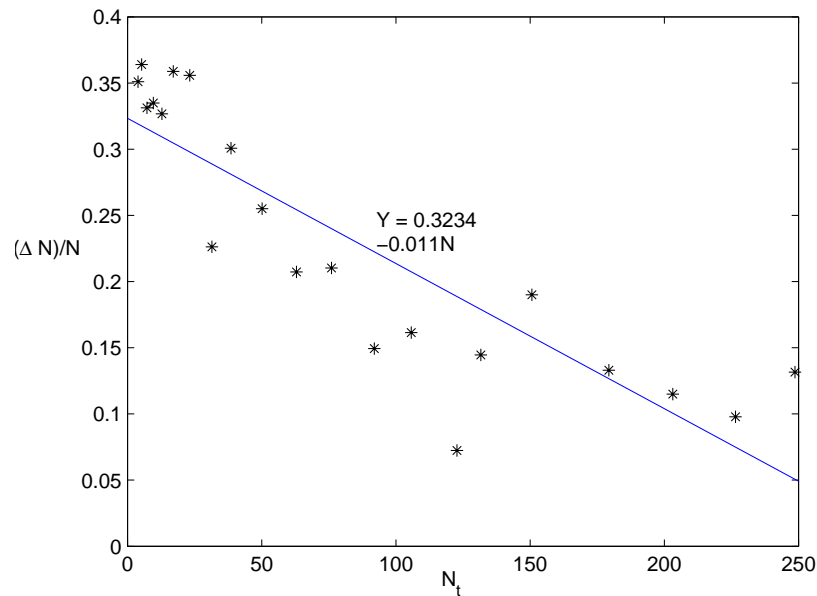


Figure 3: US Census Data Relative Growth Rate vs. Population

- (b) Plot the predicted and actual values on the same graph. How well do these predictions compare with the actual data? How does this fit compare with the Malthusian model fit of the data done in Problem (5) of Assignment #3?

*Solution:* Figure 4 Shows the US Census data from 1790 to 2000 and a logistic curve with parameters  $K \approx 294.7$  millions and  $r \approx 32.3\%$ .

Comparing this fit with the Malthusian model fit in Figure 5, we see that the logistic curve fit provides a better fit than the Malthusian one for up to 12 decades after 1790. Notice that after the eighth decade from 1790, the Malthusian model predictions begin to diverge considerably from the actual US Census values. Overall, the logistic model provides a better fit

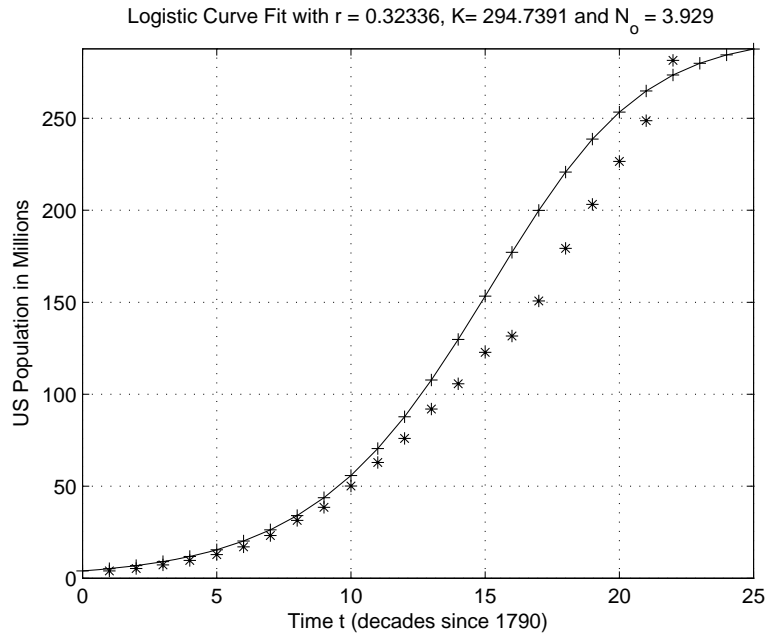


Figure 4: US Census Data and Logistic Curve Fit

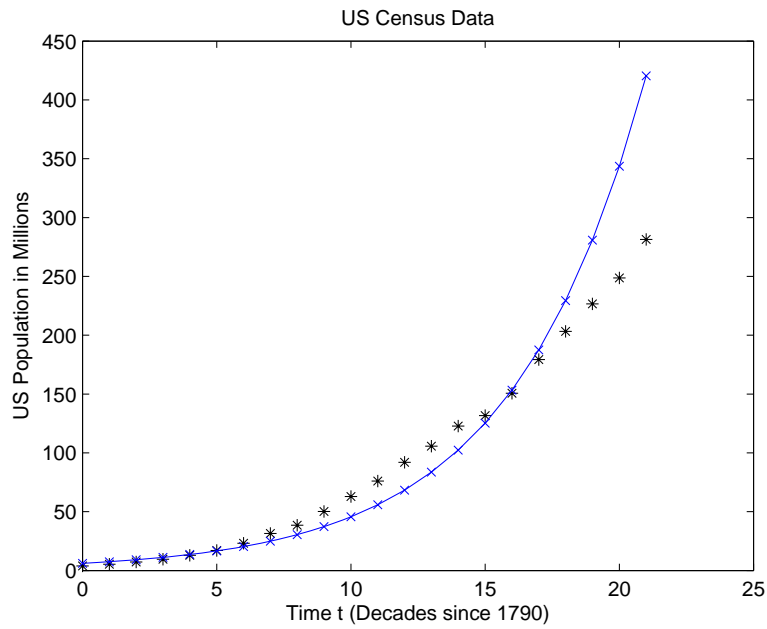


Figure 5: US Census Data and Values Predicted by Malthusian Model

- Use the procedure outlined in the previous two problems to fit a logistic curve through the *Insect Population Values* data found in Table 1.6, p. 18, in Allman and Rhodes. What are the estimated values of  $r$  and  $K$  for the insect population?

*Solution:* Figure 6 shows the plot of the relative increments versus population values for the insect population. The figure also shows the least-squares regression line. The fit is nearly perfect. (I have the suspicion that the data in Table 1.6 of Allman and Rhodes are not “real data.” Most likely they were computed using the logistic curve formula derived from the continuous logistic model .)

□

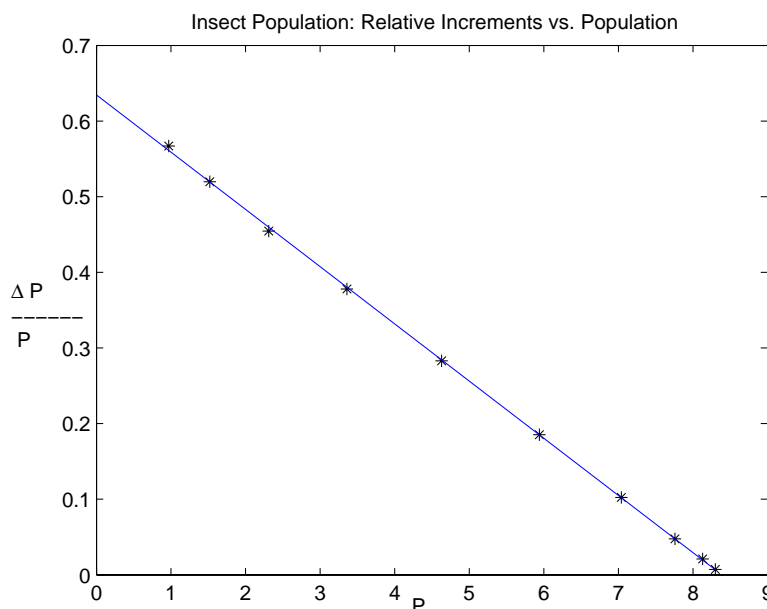


Figure 6: Plot of  $\Delta P/P$  versus  $P$  and Least-Squares Regression Line

The estimated values of  $K$  and  $r$  for the insect population are  $K = 8.39$  and  $r = 63.4\%$ . Figure 7 shows predicted values of  $P_t$ , for  $t = 1, 2, \dots, 10$ , according to the logistic model. The figure also shows the actual data values from the table. There is no difference between the two set of values. □

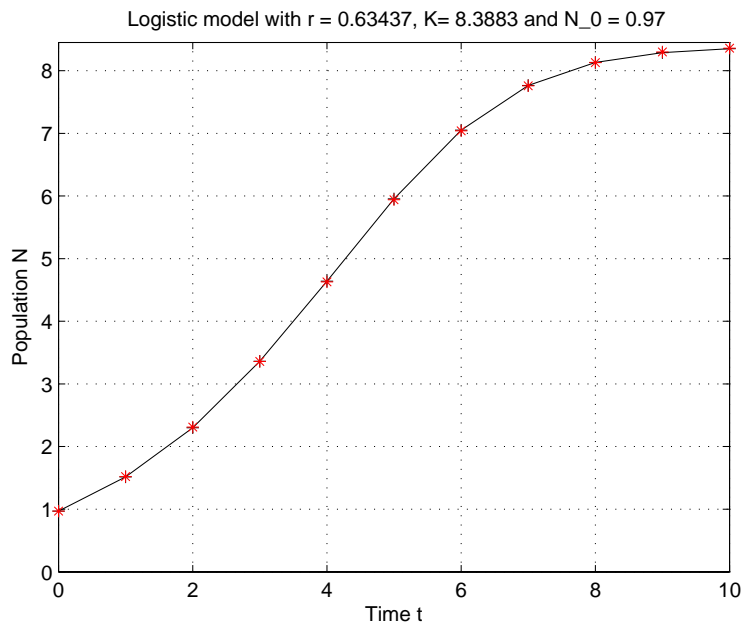


Figure 7: Insect Population: Predicted values and “actual” data