Solutions to Assignment #8

1. Suppose a bacterial colony is growing according to the (continuous) Malthusian model. If the time, t, is measured in units of *division cycle* divided by $\ln 2$, give a formula for $N(t)$, given that $N(0) = N_o$. By how much does the population increase in one unit of time? (*Note:* A *division cycle* would correspond to the doubling time.)

   Solution: Assume first the $N = N(\tau)$, where τ is measured in an arbitrary continuous time unit. Then, the solution to the Malthus differential equation

   $$\frac{dN}{d\tau} = kN,$$

   subject to $N(0) = N_o$, is given by

   $$N(t) = N_o e^{k\tau}.$$

   If T is the doubling time, then $k = \dfrac{\ln(2)}{T}$ and so

   $$N(\tau) = N_o e^{\frac{\ln(2)}{T}\tau} = N_o e^{\frac{\tau}{T/\ln(2)}}.$$

   Thus, if t counts the number of division cycles divided by $\ln(2)$, it follows that $t = \dfrac{\tau}{T/\ln(2)}$; and therefore

   $$N(t) = N_o e^t.$$

   Therefore, in one division cycle divided by $\ln(2)$, the population increases by

   $$\frac{N(1) - N(0)}{N_o} = \frac{N_o e - N_o}{N_o} = e - 1 \approx 1.718$$

   or about 172%. $\square$

2. Suppose a quantity Q varies in time according to the differential equation

$$\frac{dQ}{dt} = aQ + b, \tag{1}$$

where a and b are real constants with $a \neq 0$.

(a) Use separation of variables to find the general solution to equation (1).

Solution: Write the equation in the form $\frac{dQ}{dt} = a\left(Q + \frac{b}{a}\right)$ and separate variables to get

$$\int \frac{1}{Q + \frac{b}{a}}\, dQ = \int a\, dt,$$

which yields

$$\ln\left|Q + \frac{b}{a}\right| = at + c_1,$$

for some constant c_1. Exponentiating on both sides of the previous equation, and then solving for $Q = Q(t)$ yields

$$Q(t) = -\frac{b}{a} + Ce^{at}, \tag{2}$$

for some constant C. $\square$

(b) Find a solution to (1) satisfying $Q(0) = Q_o$.

Solution: Substituting 0 for t in equation (2), we get

$$-\frac{b}{a} + C = Q_o;$$

thus, $C = Q_o + \frac{b}{a}$ and so

$$Q(t) = -\frac{b}{a} + \left(Q_o + \frac{b}{a}\right)e^{at}. \qquad \square \tag{3}$$

(c) What does equation (1) predict about $\lim\limits_{t\to\infty} Q(t)$ if (i) $a < 0$, and (ii) if $a > 0$.

Solution: If $a < 0$, then $\lim\limits_{t\to\infty} Q(t) = -\frac{b}{a}$. On the other hand, if $a > 0$, then $\lim\limits_{t\to\infty} Q(t) = \pm\infty$, depending on whether $Q_o + \frac{b}{a}$ is positive or negative, respectively. $\square$

3. Give the equilibrium point of the differential equation (1) for $a \neq 0$. Discuss the stability type of this equilibrium point if (i) $a < 0$, and (ii) if $a > 0$. Sketch solutions starting near the equilibrium point for each of these cases.

   Solution: To find equilibrium points of the equation (1), we solve the equation

   $$aQ + b = 0,$$

   which yields $\overline{Q} = -\dfrac{b}{a}$ since $a \neq 0$. By (3), any solution of (1) satisfying $Q(0) = Q_o$ is then given by

   $$Q(t) = \overline{Q} + (Q_o - \overline{Q})e^{at}.$$

   Thus, if $a < 0$, $Q(t)$ will tend towards $\overline{Q}$ as t increases, and so $\overline{Q}$ is stable in this case. On the other hand, if $a > 0$, $Q(t)$ will tend away from $\overline{Q}$ as t increases, and so $\overline{Q}$ is unstable.

   Figure 1 shows graphs of solutions around $\overline{Q}$ for the case $b = 1$ and $a = -1$.

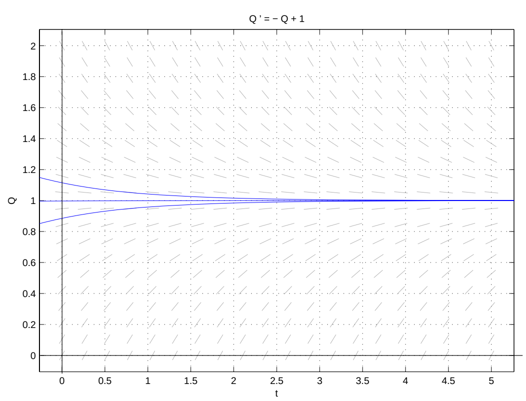

Figure 1: Sketch of solutions to (1) for $a = -1$ and $b = 1$

Figure 2 shows graphs of solutions around $\overline{Q}$ for the case $b = -1$ and $a = 1$.

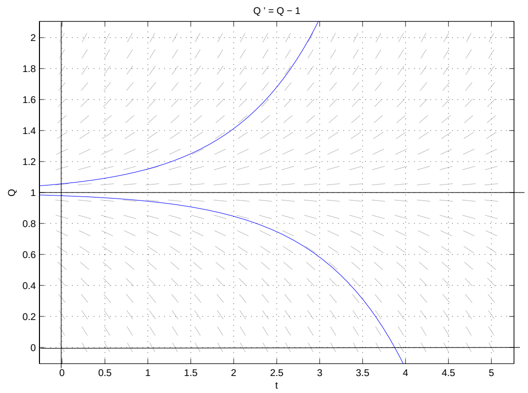

Figure 2: Sketch of solutions to (1) for $a = 1$ and $b = -1$

A Conservation Principle for a One–Compartment Model. Suppose you are tracking the amount, $Q(t)$, of a substance in some predefined space or region, known as a *compartment,* at time t. (A compartment could represent, for instance, the bloodstream in the body of a patient, and $Q(t)$ the amount of a drug present in the bloodstream at time t). If we know, or can model, the rates at which the substance enters or leaves the compartment, then the rate of the change of the substance in the compartment is determined by the differential equation:

$$\frac{dQ}{dt} = \text{Rate of substance in} - \text{Rate of substance out}, \tag{4}$$

where we are assuming that Q is a differentiable function of time.

4. Assume that the rate at which a drug leaves the bloodstream and passes into the urine is proportional to the quantity of the drug in the blood at that time.

   (a) Use the conservation principle (4) to write down a differential equation for the quantity, Q, of the drug in the blood at time, t, in hours.

   Solution: In this case, Rate of Q in $= 0$ and Rate of Q out $= kQ$, where $k > 0$ is a constant of proportionality. Thus, the conservation principle (4) implies that

   $$\frac{dQ}{dt} = -kQ. \qquad \square$$

   (b) Solve the differential equation derived in part (a) for the case in which an initial dose of Q_o is injected directly into the blood.

   Solution: $Q(t) = Q_o e^{-kt}$ for all t. $\square$

   (c) Assume that 20% of the initial dose is left in the blood after 3 hours. Write a formula for computing $Q(t)$ for any time t, in hours.

   Solution: If $Q(3) = 0.20\ Q_o$, then

   $$Q_o e^{-3k} = 0.20\ Q_o \quad \text{or} \quad e^{-3k} = \frac{1}{5}.$$

   Thus, $-k = \frac{1}{3}\ln\left(\frac{1}{5}\right)$, and so the solution is given by

   $$Q(t) = Q_o e^{\ln\left(\frac{1}{5}\right)\frac{t}{3}} = Q_o\left(\frac{1}{5}\right)^{t/3}$$

   for all t. $\square$

   (d) What percentage of the initial dosage of the drug is left in the patient's body after 6 hours?

   Solution: After 6 hours, the amount of drug present in the patient's blood is

   $$Q(6) = Q_o\left(\frac{1}{5}\right)^{6/3} = \frac{1}{25}Q_o,$$

   or 4% of the initial dose. $\square$

5. When people smoke, carbon monoxide is released into the air. Suppose that in a room of volume 60 m^3, air containing 5% carbon monoxide is introduced at a rate of 0.002 m^3/min. (This means that 5% of the volume of incoming air is carbon monoxide). Assume that the carbon monoxide mixes immediately with the air and the mixture leaves the room at the same rate as it enters.

   (a) Let $Q = Q(t)$ denote the volume (in cubic meters) of carbon monoxide in the room at any time t in minutes. Use the conservation principle (4) to write down a differential equation for Q.

   Solution: In this case,
$$\text{Rate of } Q \text{ in } = c_i F,$$
   where $c_i = 5\%$ is the concentration of carbon monoxide in the air that flows into the room at a flow rate $F = 0.002$ m^3/min; and
$$\text{Rate of } Q \text{ out } = c(t) F,$$
   where $c(t) = \dfrac{Q(t)}{V}$ is a concentration of carbon monoxide present in the room at time t. Here $V = 60$ m^3 is the (fixed) volume of the room. Thus, by the conservation principle (4),
$$\frac{dQ}{dt} = c_i F - \frac{F}{V} Q,$$
   or
$$\frac{dQ}{dt} = -\frac{F}{V}(Q - c_i V). \qquad \square$$

   (b) Give the equilibrium solution, $\overline{Q}$, to the differential equation in part (a).

   Solution: $\overline{Q} = c_i V = 3$ m^3.

   (c) Solve the differential equation in part (a) under the assumption that the there is no carbon monoxide in the room initially, and sketch the solution.

   Solution: Solve the initial value problem
$$\begin{cases} \frac{dQ}{dt} = -\frac{F}{V}(Q - c_i V) \\ Q(0) = 0. \end{cases}$$
   By separation of variables, the general solution of the equation is
$$Q(t) = \overline{Q} + C e^{-\frac{F}{V} t}.$$

The initial condition then implies that $\overline{Q} + C = 0$, and so $C = -\overline{Q}$. Thus, the solution to the initial value problem is

$$Q(t) = \overline{Q}(1 - e^{-\frac{F}{V}t}),$$

or

$$Q(t) = 3(1 - e^{-\frac{F}{V}t}),$$

in cubic meters, where $\dfrac{F}{V} = \dfrac{1}{3} \times 10^{-4}$ min^{-1}, and t is measured in minutes. Figure 3 shows the graph of this function

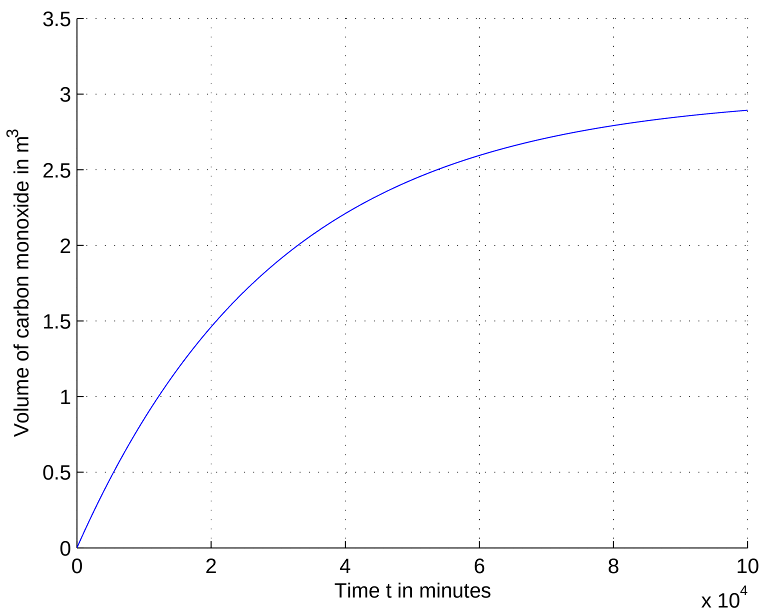

Figure 3: Graph of $Q(t) = 3(1 - e^{-\frac{F}{V}t})$

(d) Based on your solution to part (c), give the concentration, $c(t)$, of carbon monoxide in the room (in percent volume) at any time t in minutes. What happens to the value of $c(t)$ in the long run?

Solution: $c(t) = \dfrac{Q(t)}{V} = \dfrac{\overline{Q}}{V}(1 - e^{-\frac{F}{V}t}) = 5\%(1 - e^{-\frac{F}{V}t})$. Thus, as $t \to \infty$, $c(t)$ tends to 5%. $\square$

(e) Medical texts warn that exposure to air containing 0.1% carbon monoxide for some time can lead to a coma. How many hours does it take for the concentration of carbon monoxide found in part (d) to reach this level?

Solution: We want to find t such that $c(t) = 0.1\%$; that is

$$5(1 - e^{-\frac{F}{V}t}) = 0.1.$$

Solving this equation for t we get

$$t = \frac{V}{F}\ln\left(\frac{50}{49}\right) \approx 606 \text{ minutes},$$

or about 10 hours. □