Solutions to Assignment #9

1. *[The Principle of Linearized Stability].* Let $\overline{N}$ be an equilibrium point of the equation
$$\frac{dN}{dt} = g(N),$$
where g is a differentiable function with continuous derivative $g'(N)$. Show that if $g'(\overline{N}) < 0$, then $\overline{N}$ is stable, and if $g'(\overline{N}) > 0$, the $\overline{N}$ is unstable.

   Solution: Suppose that $g(\overline{N}) = 0$ and $g'(\overline{N}) < 0$. Then g is strictly decreasing in an interval around $\overline{N}$; thus, $g(N) > 0$ for $N < \overline{N}$, and $g(N) < 0$ for $N > \overline{N}$, and N in that interval. Thus, $N'(t) = g(N) > 0$ for $N < \overline{N}$ and very close to $\overline{N}$, which implies that $N(t)$ increases. It then follows that any solution of the equation that starts close to and below $\overline{N}$ will tend towards it. On the other hand, since $N'(t) = g(N) < 0$ for $N > \overline{N}$ and close to it, any solution of the equation that begins above $\overline{N}$, and very close to it, will decrease towards it as t increases. We therefore conclude that $\overline{N}$ is stable.

   A similar argument yields instability in the case $g'(N) > 0$. □

2. Give examples of the differential equation
$$\frac{dN}{dt} = g(N)$$
with an equilibrium point $\overline{N}$ such that $g'(\overline{N}) = 0$, and for which

   (a) $\overline{N}$ is stable,
   Solution: Use the example $g(N) = -N^3$. Then, $\overline{N} = 0$ is an equilibrium point with $g'(\overline{N}) = 0$. Observe that $g(N)$ is strictly decreasing in this case, and so the argument used in the previous problem yields stability in this case. □

   (b) $\overline{N}$ is unstable, and
   Solution: Use the example $g(N) = N^3$. Then, $\overline{N} = 0$ is an equilibrium point with $g'(\overline{N}) = 0$. However, in this case, $g(N)$ is strictly increasing and so we get instability. Here, we could have also used $g(N) = N^2$. In this case, $g(N) > 0$ for all $N \neq \overline{N}$. Thus, $N'(t) > 0$ for any solution near 0, and so $N(t)$ is always increasing; in particular, any solution that starts close to 0 and above it will tend away from 0. □

3. Show that the initial value problem (IVP)

$$\begin{cases} \dfrac{dN}{dt} & = & rN\left(1-\dfrac{N}{K}\right) \\ \\ N(0) & = & N_o, \end{cases}$$

where $0 < N_o < K$, has solution that is defined for all real values of t. Compute

$$\lim_{t\to-\infty} N(t) \quad \text{and} \quad \lim_{t\to+\infty} N(t).$$

Solution: The solution to the IVP is given by

$$N(t) = \frac{N_oK}{N_o + (K-N_o)e^{-rt}}$$

and this is defined for all t where the denominator is not zero. Observe that if $0 < N_o < K$, then $K - N_o > 0$ and so

$$N_o + (K-N_o)e^{-rt} > N_o > 0 \quad \text{for all } t \in \mathbf{R}.$$

That is, the denominator in the expression defining $N(t)$ is never zero, and so $N(t)$ is defined for all $t \in \mathbf{R}$.

Since $r > 0$, $e^{-rt} \to 0$ as $t \to \infty$, and $e^{-rt} \to +\infty$ as $t \to -\infty$, thus

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{N_oK}{N_o + (K-N_o)e^{-rt}} = K$$

and

$$\lim_{t\to-\infty} N(t) = \lim_{t\to-\infty} \frac{N_oK}{N_o + (K-N_o)e^{-rt}} = 0. \qquad \Box$$

4. *(Population "Super–Explosion").* If the *per capita* growth is assumed to be proportional to the population density, N (that is, if

$$\frac{1}{N}\,\frac{dN}{dt} = kN$$

for some constant or proportionality $k > 0$), we obtain the model

$$\frac{dN}{dt} = kN^2. \tag{1}$$

(a) Use separation of variables to solve equation (1) subject to the initial condition $N(0) = 1$.

Solution: $\displaystyle\int \frac{1}{N^2}dN = \int k\,dt$ implies that $-\dfrac{1}{N} = kt + c$ for some constant c. The initial condition then yields that $c = -1$, and so

$$\frac{1}{N} = 1 - kt,$$

or

$$N(t) = \frac{1}{1-kt}. \qquad \square$$

(b) Show that the solution obtained in part (a) ceases to exist at some (finite) time t_1. What is the value of t_1?

Solution: The solution obtained in the previous part fails to be defined when $1 - kt =$ or $t = 1/k$. Thus $t_1 = \dfrac{1}{k}$. $\square$

(c) What happens to the solution as t tends to t_1 from the left?

Solution: $\displaystyle\lim_{t\to t_1^-} N(t) = \lim_{t\to t_1^-} \frac{1}{1-kt} = +\infty.$

5. The IVP

$$\begin{cases} \frac{dN}{dt} = \sqrt{N} \\ N(0) = 0, \end{cases}$$

has the constant function 0 as a solution. Use separation of variables to compute another solution to the IVP different from the 0–solution. Why doesn't this contradict the *Local Existence and Uniqueness Theorem?*

Solution: $\int \frac{1}{\sqrt{N}} dN = \int dt$ implies that $2\sqrt{N} = t + c$ for some constant c. The initial condition then yields that $c = 0$, and so

$$2\sqrt{N} = t,$$

from which we get that

$$N(t) = \frac{1}{4}t^2$$

is another solution to the IVP. This does not contradict the *Local Existence and Uniqueness Theorem* because the function $g(N) = \sqrt{N}$ fails to be differentiable at $N = 0$, and so the theorem does not apply. $\square$