

## Solutions to Exam 1

1. Consider the difference equation

$$\Delta N = aN,$$

where  $a$  is a nonzero parameter.

- (a) Give an interpretation of the equation as a model for population growth.

**Answer:** This model assumes that the per-capita growth rate of the population is constant.  $\square$

- (b) Solve the equation given that  $N_o$  is known.

**Solution:** Write the equation in the form

$$N_{t+1} = (1 + a)N_t$$

to obtain that

$$N_t = (1 + a)^t N_o \quad \text{for } t = 0, 1, 2, \dots$$

$\square$

- (c) Find equilibrium point(s) and test for stability. Which values of  $a$  yield stability?

**Solution:** The only equilibrium point is  $N^* = 0$  since  $a \neq 0$ . It is stable if  $|1 + a| < 1$ , or  $-2 < a < 0$ .  $\square$

2. The difference equation

$$\Delta N = rN(1 - N) - EN$$

provides a discrete model for a population that is growing logistically but is also being harvested at a rate proportional to the number of individuals present at a given time. The constant of proportionality  $E$  is called the *harvesting effort*. The parameters  $r$  and  $E$  are assumed to be positive.

- (a) Write the model in the form  $N_{t+1} = f(N_t)$  and give the fixed points of  $f$ .

**Solution:**  $N_{t+1} = N_t + rN_t(1 - N_t) - EN_t$ , so that

$$f(x) = x + rx(1 - x) - Ex.$$

The fixed points of  $f$  are the solutions to  $f(x) = x$ , or

$$N_1^* = 0 \quad \text{and} \quad N_2^* = 1 - \frac{E}{r}.$$

□

- (b) Find conditions on the parameters  $r$  and  $E$  that will ensure that the model will have a non-negative steady state. What does this say about the harvesting effort in terms of the intrinsic growth rate  $r$ ?

**Solution:** For a non-negative steady state, we require that

$$E \leq r;$$

that is, the harvesting effort cannot exceed the intrinsic growth rate of the population. □

- (c) Use the Principle of Linearized Stability to determine conditions on  $r$  and  $E$  that will guarantee that the nonzero equilibrium point found in the previous part is stable.

**Solution:** Compute  $f'(N_2^*)$ , where

$$f'(x) = 1 + r - E - 2rx.$$

Thus,

$$f'(N_2^*) = 1 + r - E - 2r \left(1 - \frac{E}{r}\right) = 1 - r + E.$$

For stability we then require that

$$|1 - r + E| < 1,$$

or

$$E < r < 2 + E.$$

□

- (d) What does the model predict if  $E = r$ ?

**Answer:** If  $E = r$ , then the equation becomes

$$\Delta N = -rN^2.$$

Thus,  $\Delta N > 0$  for all population densities  $N > 0$ ; which implies that the population will always be decreasing. This model therefore predicts extinction for  $E = r$ .  $\square$

3. Figure 1 shows the graph of  $N_{t+1} = f(N_t)$ , where  $f(x) = xe^{r(1-x)}$  for  $r = 2.5$ , as well as the graph of the line  $N_{t+1} = N_t$ .

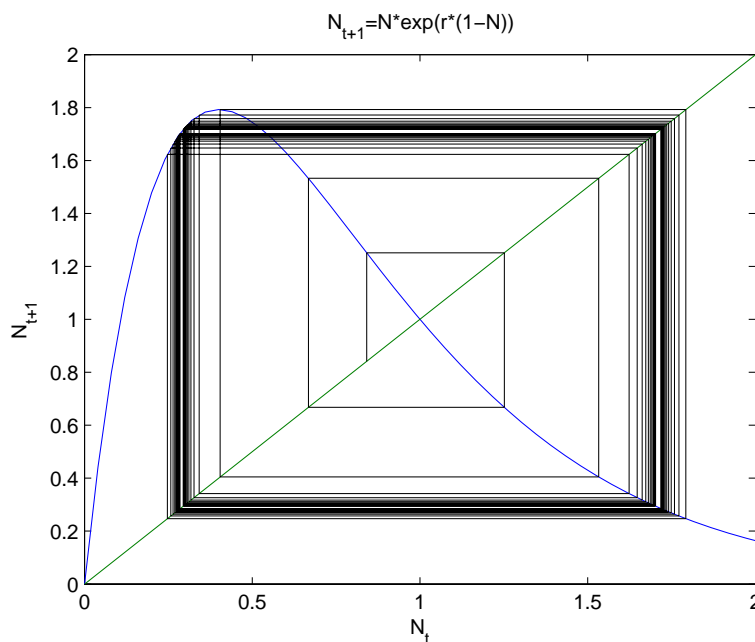


Figure 1: Cobweb diagram for  $f(x) = xe^{r(1-x)}$  with  $r = 2.5$

- (a) Give the two equilibrium points of the difference equation  $N_{t+1} = f(N_t)$  and determine their stability properties using the cobweb diagram.

**Answer:** The equilibrium points are  $N_1^* = 0$  and  $N_2^* = 1$ , and they are both unstable.  $\square$

- (b) Use the Principle of Linearized stability to verify your answers in the previous part.

**Solution:** Compute the derivative of  $f$  to get

$$f'(x) = (1 - rx)e^{r(1-x)}.$$

Testing for stability at  $N_1^*$ , we get

$$f'(0) = e^r > 1,$$

and therefore  $N_1^*$  is unstable, by the Principle of Linearized Stability.

Testing for stability at  $N_2^* = 1$  we have

$$f'(1) = 1 - r.$$

Thus, since  $r = 2.5$ ,  $f'(1) = -1.5$ , and so  $|f'(1)| > 1$ . Consequently,  $N_2^*$  is also unstable by the Principle of Linearized Stability.  $\square$

- (c) Describe the long-term behavior of solutions to the difference equation whose initial values,  $N_o$ , range from 0 to 2.

**Answer:** Solutions will oscillate.  $\square$