Solutions to Assignment #10

1. Let $U = \mathbb{R}^n\backslash\{\mathbf{0}\} = \{v \in \mathbb{R}^n \mid v \neq \mathbf{0}\}$ and define $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(v) = \|v\| \quad \text{for all} \quad v \in \mathbb{R}.$$

(a) Prove that f is differentiable on U.

Solution: For $v = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, write

$$f(v) = f(x_1, x_2, \ldots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

and observe that if $(x_1, x_2, \ldots, x_n) \in U$, then $x_1^2+x_2^2+\cdots+x_n^2 \neq 0$ so that the partial derivatives

$$\frac{\partial f}{\partial x_j}(x_1, x_2, \ldots, x_n) = \frac{x_j}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}, \quad j = 1, 2, \ldots, n,$$

exist in U and are continuous there. Therefore, f is a C^1 map in U and it is therefore differentiable in U. □

(b) Prove that f is not differentiable at the origin in $\mathbb{R}^n$.

Solution: Arguing by contradiction, assume that f is differentiable at the origin. Then, there exists a linear transformation, $T\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$f(w) = T(w) + E_o(w), \tag{1}$$

for $\|w\|$ small, where

$$\lim_{\|w\|\to 0} \frac{\|E_o(w)\|}{\|w\|} = 0. \tag{2}$$

Take $w = te_j$, where e_j is one of the standard basis vectors. It then follows from (1) that

$$|t| = tT(e_j) + E_o(te_j),$$

for $t \in \mathbb{R}$ with $|t|$ sufficiently small. Thus, if $t \neq 0$ and $|t|$ is sufficiently small,

$$\frac{|t|}{t} = T(e_j) + \frac{1}{t}E_o(te_j).$$

Observe that, by (2),

$$\lim_{t\to 0}\frac{1}{t}E_o(te_j) = \mathbf{0}.$$

Hence,

$$\lim_{t\to 0}\frac{|t|}{t} = T(e_j),$$

which is impossible since $\displaystyle\lim_{t\to 0}\frac{|t|}{t}$ does not exist. Consequently, $f(v) = \|v\|$ is not differentiable at the origin. □

2. Let I be and open interval of real numbers, and suppose that $\sigma\colon I \to \mathbb{R}^n$ is a differentiable path satisfying $\sigma(t) \neq \mathbf{0}$ for all $t \in I$. Show that the function $g\colon I \to \mathbb{R}$ defined by $g(t) = \|\sigma(t)\|$ for all $t \in I$ is differentiable on I and compute its derivative.

    ***Solution**:* Let f be as defined in Problem (1a) and observe that

    $$g = f \circ \sigma.$$

    Note that, since $\sigma(t) \neq \mathbf{0}$ for all $t \in I$, $\sigma(I) \subseteq U$, where U is as defined in Problem 1a. Consequently, by the result of part (a) in Problem 1a, g is differentiable on I, by the Chair Rule, and its derivative is given by

    $$Dg(t) = Df(\sigma(t))D\sigma(t) = \nabla f(\sigma(t)) \cdot \sigma'(t), \quad \text{for all } \ t \in I,$$

    where

    $$\nabla f(v) = \frac{1}{\|v\|}v \quad \text{for all } \ v \in U.$$

    We then have that

    $$g'(t) = \frac{1}{\|\sigma(t)\|}\sigma(t) \cdot \sigma'(t), \quad \text{for all } \ t \in I.$$

    □

3. Let I be an open interval of real numbers and U ba an open subset of $\mathbb{R}^n$. Suppose that $\sigma\colon I \to \mathbb{R}^n$ is a differentiable path and that $f\colon U \to \mathbb{R}$ is a differentiable scalar field. Assume also that the image of I under σ, $\sigma(I)$, is contained in U. Suppose also that the derivative of the path σ satisfies

$$\sigma'(t) = -\nabla f(\sigma(t)) \quad \text{for all} \quad t \in I.$$

Show that if the gradient of f along the path σ is never zero, then f decreases along the path as t increases.

Suggestion: Use the Chain Rule to compute the derivative of $f(\sigma(t))$.

***Solution**:* Using the Chain Rule to compute the derivative of $f(\sigma(t))$ we obtain that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(f(\sigma(t))) &= \nabla f(\sigma(t)) \cdot \sigma'(t) \\ &= -\nabla f(\sigma(t)) \cdot \nabla f(\sigma(t)) \\ &= -\|\nabla f(\sigma(t))\|^2. \end{aligned}$$

It then follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(\sigma(t))) < 0 \quad \text{for all} \quad t \in I,$$

and therefore $f(\sigma(t))$ decreases as t increases. □

4. A set $U \subseteq \mathbb{R}^n$ is said to be **path connected** iff for any vectors x and y in U, there exists a differentiable path $\sigma\colon [0,1] \to \mathbb{R}^n$ such that $\sigma(0) = x$, $\sigma(1) = y$ and $\sigma(t) \in U$ for all $t \in [0,1]$; i.e., any two elements in U can be connected by a differentiable path whose image is entirely contained in U.

Suppose that U is an open, path connected subset of $\mathbb{R}^n$. Let $f\colon U \to \mathbb{R}$ be a differentiable scalar field such that $\nabla f(x)$ is the zero vector for all $x \in U$. Prove that f must be constant.

***Solution**:* Fix a point x_o in U. Then, for any $x \in U$, there exists a differentiable path $\sigma\colon [0,1] \to \mathbb{R}^n$ such that $\sigma(0) = x_o$, $\sigma(1) = x$, and $\sigma(t) \in U$ for all $t \in [0,1]$. Then $f(\sigma(t))$ for all $t \in I$ defines a differentiable function on $(0,1)$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(\sigma(t))) = \nabla f(\sigma(t)) \cdot \sigma'(t) \quad \text{for all} \quad t \in (0,1).$$

Thus, since $\nabla f(y) = \mathbf{0}$ for all $y \in U$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(\sigma(t))) = 0 \quad \text{for all} \quad t \in (0,1),$$

which implies that $f(\sigma(t))$ is constant on $(0,1)$. It then follows, by continuity, that

$$f(\sigma(1)) = f(\sigma(0)),$$

or

$$f(x) = f(x_o).$$

Since this is true for every $x \in U$, we conclude that f is constant on U. □

5. *(Exercises 2 and 4 on page 207 in the text).*

   (a) *(Exercise 2 on pg. 207)* Let x and y be functions of u and v: $x = x(u,v)$, $y = y(u,v)$, and let $f(x,y)$ be a scalar field. Find $\partial f/\partial u$ and $\partial f/\partial v$ in terms of $\partial f/\partial x$, $\partial f/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$.

   ***Solution**:* Apply the Chain Rule to $f(x(u,v), y(u,v))$ to get that

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

   and

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

   □

   (b) *(Exercise 4 on pg. 207)* For f x and y as in Exercise 2, express $\partial^2 f/\partial u^2$ in terms of the partial derivatives of f with respect to x and y and the partial derivatives of x and y with respect to u. Assume that

$$\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x.$$

Solution: Apply the Product Rule and the Chain Rule to get

$$
\begin{aligned}
\frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\right) \\
&= \frac{\partial f}{\partial x}\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial u}\right) + \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial u} \\
&\quad + \frac{\partial f}{\partial y}\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u}\right) + \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial u} \\
&= \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial u^2} + \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u}\right)\frac{\partial x}{\partial u} \\
&\quad + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial u^2} + \left(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial u}\right)\frac{\partial y}{\partial u} \\
&= \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u}\frac{\partial y}{\partial u} \\
&\quad + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2.
\end{aligned}
$$

□