Solutions to Assignment #11

1. Let I denote an open interval in $\mathbb{R}$, and $\sigma\colon I \to \mathbb{R}^n$ be a C^1 path. For fixed $a \in I$, define

$$s(t) = \int_a^t \|\sigma'(\tau)\|\, \mathrm{d}\tau \quad \text{for all} \ \ t \in I.$$

Show that s is differentiable and compute $s'(t)$ for all $t \in I$.

Solution: Since the σ is a C^1 path, the map $t \mapsto \|\sigma'(t)\|$ is continuous on I. Therefore, by the Fundamental Theorem of Calculus, $s(t)$ is differentiable and

$$s'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \|\sigma'(\tau)\|\, \mathrm{d}\tau = \|\sigma'(t)\| \quad \text{for all} \ \ t \in I.$$

$\square$

2. Let σ and s be as defined in the previous problem. Suppose, in addition, that $\sigma'(t)$ is never the zero vector for all t in I. Show that s is a strictly increasing function of t and that it is, therefore, one–to–one.

Solution: From the previous problem,

$$s'(t) = \|\sigma'(t)\| \quad \text{for all} \ \ t \in I,$$

so that, since $\sigma'(t)$ is never the zero vector for all t in I, $s'(t) > 0$ for all $t \in I$. It then follows that s is a strictly increasing function of t and, therefore, it is a one–to–one map. $\square$

3. Let σ and s be as defined in Problem 1. We can re–parameterize σ by using s as a parameter. We therefore obtain $\sigma(s)$, where s is the *arc length* parameter.

Differentiate the expression

$$\sigma(s(t)) = \sigma(t)$$

with respect to t using the Chain Rule. Conclude that, if $\sigma'(t)$ is never the zero vector for all t in I, then $\sigma'(s)$ is always a unit vector.

The vector $\sigma'(s)$ is called the *unit tangent vector* to the path σ.

Solution: Differentiate

$$\sigma(s(t)) = \sigma(t)$$

with respect to t to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma(s(t)) = \sigma'(t);$$

thus, by the Chain Rule,

$$\sigma'(s)s'(t) = \sigma'(t),$$

or

$$\sigma'(s)\|\sigma'(t)\| = \sigma'(t).$$

Since $\|\sigma'(t)\| \neq 0$ for all $t \in I$, we have that

$$\sigma'(s) = \frac{1}{\|\sigma'(t)\|}\sigma'(t),$$

and therefore $\sigma'(s)$ is a unit vector. $\square$

4. For a and b, positive real numbers, the expression

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

defines an ellipse in the xy–plane $\mathbb{R}^2$.

Sketch the ellipse, give a parametrization for it, and set up the integral that yields its arc length.

Solution: A sketch of the ellipse for the case $b < a$ is shown in Figure 1.

The path

$$\sigma(t) = (a\cos t, b\sin t) \quad \text{for all } \ t \in [0, 2\pi]$$

is a C^1 parametrization of the ellipse. The arc length of the ellipse is then given by

$$\int_0^{2\pi} \|\sigma'(t)\| \ \mathrm{d}t,$$

where

$$\sigma'(t) = (-a\sin t, b\cos t) \quad \text{for all } \ t \in \mathbb{R},$$

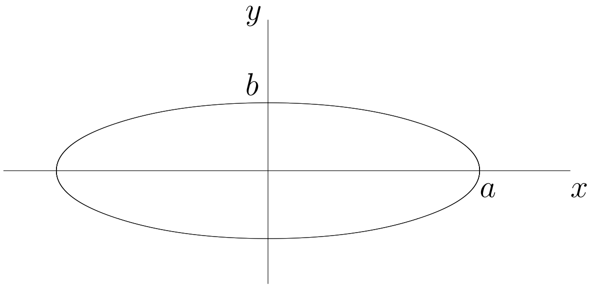

Figure 1: Sketch of ellipse for $b < a$

Thus

$$\int_0^{2\pi} \|\sigma'(t)\| \, \mathrm{d}t = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \, \mathrm{d}t.$$

$\square$

5. Let $\sigma\colon [0,\pi] \to \mathbb{R}^3$ be defined by $\sigma(t) = t\,\widehat{i} + t\sin t\,\widehat{j} + t\cos t\,\widehat{k}$ for all $t \in [0,\pi]$. Compute the arc length of the curve parametrized by σ.

***Solution**:* Let C denote the curve parametrized by σ; then,

$$\ell(C) = \int_0^{\pi} \|\sigma'(t)\| \, \mathrm{d}t,$$

where

$$\sigma'(t) = \widehat{i} + (\sin t + t\cos t)\,\widehat{j} + (\cos t - t\sin t)\,\widehat{k} \quad \text{for all} \quad t \in \mathbb{R},$$

and therefore

$$\begin{aligned} \|\sigma'(t)\| &= \sqrt{1 + (\sin t + t\cos t)^2 + (\cos t - t\sin t)^2} \\ &= \sqrt{2 + t^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \ell(C) &= \int_0^{\pi} \sqrt{2 + t^2} \, \mathrm{d}t \\ &= \left[\frac{t}{2}\sqrt{2 + t^2} + \ln|t + \sqrt{2 + t^2}| \right]_0^{\pi} \\ &= \frac{\pi}{2}\sqrt{2 + \pi^2} + \ln(\pi + \sqrt{2 + \pi^2}) - \frac{1}{2}\ln 2. \end{aligned}$$

□