Solutions to Assignment #1

1. Let $\vec{v}_1 = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$.

   (a) Give the parametric equations of the line through the point $P\colon (0,4,7)$ in the direction of the vector $\vec{v}_1$.

   (b) Give the equation of the plane through the point $P\colon (0,4,7)$ and perpendicular to a direction which is perpendicular to both vectors $\vec{v}_1$ and $\vec{v}_2$.

   Solution:

   (a) $\begin{cases} x = -t \\ y = 4+2t \\ z = 7-2t \end{cases}$

   (b) The plane is given by

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 7 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} + s\begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix},\ t,s \in \mathbb{R} \right\}.$$

   This leads to the three equations

$$\begin{cases} x = -t+3s \\ y = 4+2t-5s \\ z = 7-2t+4s \end{cases}$$

   which can be expressed as a system of linear equations in which the unknowns are t and s:

$$\begin{cases} -t+3s = x \\ 2t-5s = y-4 \\ -2t+4s = z-7 \end{cases}$$

   We can use Gaussian elimination to determine conditions on x, y and z for which it is possible to solve this system for t and s:

$$\left(\begin{array}{rr|c} -1 & 3 & x \\ 2 & -5 & y-4 \\ -2 & 4 & z-7 \end{array}\right) \to \left(\begin{array}{rr|c} 1 & -3 & -x \\ 2 & -5 & y-4 \\ -2 & 4 & z-7 \end{array}\right) \to$$

$$\begin{pmatrix} 1 & -3 & | & -x \\ 0 & 1 & | & 2x+(y-4) \\ 0 & -2 & | & -2x+(z-7) \end{pmatrix} \to \begin{pmatrix} 1 & -3 & | & -x \\ 0 & 1 & | & 2x+(y-4) \\ 0 & 0 & | & 2x+2(y-4)+(z-7) \end{pmatrix}$$

Thus, for the system to have a solution, we must have that

$$2x + 2(y-4) + (z-7) = 0.$$

This is the equation of the plane, which may be simplified to

$$2x + 2y + z = 15.$$

□

2. The following give parametric equations to two lines in $\mathbb{R}^3$:

$$\begin{cases} x = -1+4t \\ y = -7t \\ z = 2-t \end{cases} \qquad \begin{cases} x = -1+s \\ y = 2-s \\ z = 2s \end{cases}$$

Determine if the two lines ever meet. Justify your answer. If the lines do meet, give the equation of the plane that contains both lines.

***Solution**:* Suppose there are values for t and s that yield the same point in space. Then,

$$\begin{aligned} -1+4t &= -1+s \\ -7t &= 2-s \\ 2-t &= 2s \end{aligned}$$

from which we get the system

$$\begin{cases} 4t - s = 0 \\ -7t + s = 2 \\ -t - 2s = -2 \end{cases}$$

Solving this system by Gaussian elimination we get

$$\begin{pmatrix} 4 & -1 & | & 0 \\ -7 & 1 & | & 2 \\ -1 & -2 & | & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & | & 2 \\ -7 & 1 & | & 2 \\ 4 & -1 & | & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 2 & | & 2 \\ 0 & 15 & | & 16 \\ 0 & -9 & | & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & | & 2 \\ 0 & 1 & | & \frac{16}{15} \\ 0 & 0 & | & \frac{8}{5} \end{pmatrix}$$

The last row yields a contradiction. Thus, the system has no solutions and therefore the lines do not meet. □

3. The following give parametric equations to two lines in $\mathbb{R}^3$:

$$\begin{cases} x = 2 + 4t \\ y = -1 - 7t \\ z = 2 - t \end{cases} \qquad \begin{cases} x = s \\ y = 1 - s \\ z = -2 + 2s \end{cases}$$

Determine if the two lines ever meet. Justify your answer. If the lines do meet, give the equation of the plane that contains both lines.

***Solution**:* Proceeding as in the previous problem (or by inspection), we find that when $t = 0$ and $s = 2$ the two lines yield the same point $(2, -1, 2)$. Thus, the two lines do meet at that point.

The two lines are contained in a plane through that point spanned by the direction vectors of the two lines

$$\begin{pmatrix} 4 \\ -7 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Thus, the plane is given by

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 4 \\ -7 \\ -1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \ t, s \in \mathbb{R} \right\}.$$

Proceeding as in the solution to problem 1, we are lead to the solving the system that yields the augmented matrix

$$\begin{pmatrix} 4 & 1 & | & x - 2 \\ -7 & -1 & | & y + 1 \\ -1 & 2 & | & z - 2 \end{pmatrix}$$

Performing Gaussian elimination yields

$$\begin{pmatrix} 1 & -2 & | & -(2 - 2) \\ 0 & 1 & | & -\frac{1}{15}(y + 1) + \frac{7}{15}(z - 2) \\ 0 & 0 & | & (x - 2) + \frac{3}{5}(y + 1) - \frac{1}{5}(z - 2) \end{pmatrix}$$

It then follows that the equation of the plane containing the two lines is

$$(x-2)+\frac{3}{5}(y+1)-\frac{1}{5}(z-2)=0$$

or

$$5(x-2)+3(y+1)-(z-2)=0$$

or

$$5x+3y-z=5.$$

□

4. The vectors $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$ can span a line, a plane or the entire three dimensional space $\mathbb{R}^3$. Give the equation of the geometric object which they span.

***Solution**:* Place the vectors as rows in the matrix

$$\begin{pmatrix} -1 & 1 & 2 \\ 1 & 0 & -1 \\ 3 & 4 & 1 \end{pmatrix}$$

and perform elementary row operations, keeping a record of the changes, to obtain that

$$\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

where the last row was obtained by $-4\vec{v}_1-7\vec{v}_2+\vec{v}_3$. It then follows that the three vectors are linearly dependent with the third one expressed as

$$v_3 = 4\vec{v}_1 + 7\vec{v}_2,$$

the first two vectors being linearly independent. Consequently, the three vectors generate a plane spanned by the first two vectors. Proceeding as in problems 1 and 3, we find that the equation of the plane is

$$x-y+z=0.$$

□

5. *(Exercises 10 and 11 on page 50 in the text.)* Consider the plane whose equation is

$$x - 4y + 7z = 3$$

10. Find a vector perpendicular to this plane.

    Solution: $n = \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$. □

11. Find two vectors that span this plane.

    Solution: The points $P_o(3, 0, 0)$, $P_1(0, -3/4, 0)$ and $P_2(0, 0, 3/7)$ are three point on the plane. The vectors

    $$\vec{v}_1 = \overrightarrow{P_oP_1} = \begin{pmatrix} -3 \\ -\frac{3}{4} \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \overrightarrow{P_oP_2} = \begin{pmatrix} -3 \\ 0 \\ \frac{3}{7} \end{pmatrix}$$

    lie on the plane and they also generate it. □