Solutions to Assignment #2

1. Recall that the dot product, or inner product, of two vectors in $\mathbb{R}^n$ is symmetric, bi–linear and positive definite; that is, for vectors v, v_1, v_2 and w in $\mathbb{R}^n$,

   (i) $v \cdot w = w \cdot v$

   (ii) $(c_1 v_1 + c_2 v_2) \cdot w = c_1 v_1 \cdot w + c_2 v_2 \cdot w$, and

   (iii) $v \cdot v \geqslant 0$ for all $v \in \mathbb{R}^n$ and $v \cdot v = 0$ if and only if v is the zero vector.

   Use these properties of the the inner product in $\mathbb{R}^n$ to derive the following properties of the norm $\|\cdot\|$ in $\mathbb{R}^n$, where

   $$\|v\| = \sqrt{v \cdot v} \quad \text{for all vectors } \ v \in \mathbb{R}^n.$$

   (a) $\|v\| \geqslant 0$ for all $v \in \mathbb{R}^n$ and $\|v\| = 0$ if and only if $v = \vec{0}$.

   Solution: By the definition, $\|v\| = \sqrt{v \cdot v}$, of the norm and the positive definiteness of the dot product, we see that $\|v\| \geqslant 0$ for all $v \in \mathbb{R}^n$. Furthermore, $\|v\| = 0$ if and only if $v = \vec{0}$. □

   (b) For a scalar c, $\|cv\| = |c|\|v\|$.

   Solution: Write $\|cv\|^2 = (cv) \cdot (cv) = c^2 v \cdot v$, by the bi–linearity of the dot product. Thus,

   $$\|cv\|^2 = (cv) \cdot (cv) = c^2 \|v\|^2.$$

   Taken square roots on both sides we get

   $$\|cv\| = \sqrt{c^2}\|v\| = |c|\|v\|.$$

   □

2. Recall the Cauchy-Schwarz inequality: For any vectors v and w in $\mathbb{R}^n$,

   $$|v \cdot w| \leqslant \|v\|\|w\|.$$

   Use this inequality to derive the triangle inequality: For any vectors v and w in $\mathbb{R}^n$,

   $$\|v + w\| \leqslant \|v\| + \|w\|.$$

   (*Suggestion:* Start with the expression $\|v + w\|^2$ and use the properties of the inner product to simplify it.)

***Solution**:* Expand $\|v+w\|^2$ to get, using the properties of the dot product, that

$$\begin{aligned}\|v+w\|^2 &= (v+w)\cdot(v+w)\\ &= v\cdot v + v\cdot w + w\cdot v + w\cdot w\\ &= \|v\|^2 + 2v\cdot w + \|w\|^2\\ &\leqslant \|v\|^2 + 2|v\cdot w| + \|w\|^2.\end{aligned}$$

It then follows by the Cauchy–Schwarz inequality that

$$\|v+w\|^2 \leqslant \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2.$$

Taking square roots yields the triangle inequality

$$\|v+w\| \leqslant \|v\| + \|w\|.$$

□

3. Given two non–zero vectors v and w in $\mathbb{R}^n$, the cosine of the angle, θ, between the vectors can be defined by

$$\cos\theta = \frac{v\cdot w}{\|v\|\|w\|}.$$

Use the Cauchy-Schwarz inequality to justify why this definition makes sense.

***Solution**:* By the Cauchy–Schwarz inequality

$$\left|\frac{v\cdot w}{\|v\|\|w\|}\right| = \frac{|v\cdot w|}{\|v\|\|w\|} \leqslant 1.$$

Thus it makes sense to say that $\dfrac{v\cdot w}{\|v\|\|w\|}$ is the cosine of some angle.

□

4. Two vectors v and w in $\mathbb{R}^n$ are said to be *orthogonal* or perpendicular, if and only if $v\cdot w = 0$.

Show that if v and w are orthogonal, then

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2.$$

Give a geometric interpretation of this result in two–dimensional Euclidean space.

Solution: Expand $\|v+w\|^2$ to get, using the properties of the dot product, that

$$\begin{aligned}\|v+w\|^2 &= (v+w)\cdot(v+w)\\ &= v\cdot v + v\cdot w + w\cdot v + w\cdot w\\ &= \|v\|^2 + 2v\cdot w + \|w\|^2\end{aligned}$$

Thus, since $v\cdot w = 0$, it follows that

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2.$$

This is the Pythagorean Theorem. □

5. A vector u in $\mathbb{R}^n$ is said to be a unit vector if and only if $\|u\| = 1$. Let u be a unit vector in $\mathbb{R}^n$ and v be any vector in $\mathbb{R}^n$.

(a) Give the parametric equation of the line through origin in the direction of u.

Solution: If we let $L(t)$ denote a point of the line

$$L(t) = tu$$

gives the parametric equation of the line. □

(b) Let $f(t) = \|v - tu\|^2$ for all $t \in \mathbb{R}^n$. Explain why this function gives the square of the distance from the point at v to a point on the line through the origin in the direction of u.

Solution: The norm of the difference of two vectors gives the distance between the vectors. □

(c) Show that $f(t)$ is minimized when $t = v\cdot u$.

Solution: Expand $f(t)$ by using the properties of the dot product to get

$$\begin{aligned}f(t) &= \|v - tu\|^2\\ &= (v-tu)\cdot(v-tu)\\ &= v\cdot v - v\cdot(tu) - (tu)\cdot v + (tu)\cdot(tu)\\ &= \|v\|^2 - 2tv\cdot u + t^2\|u\|^2\\ &= \|v\|^2 - 2tv\cdot u + t^2\end{aligned}$$

since u is a unit vector. We then see that $f(t)$ is a quadratic polynomial in t. Since,

$$f'(t) = -2v \cdot u + 2t$$

and

$$f'(t) = 2 > 0,$$

we see that $f(t)$ has a minimum when $f'(t) = 0$, or when $t = v \cdot u$.
□