

Solutions to Assignment #3

1. The vectors $v_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ span a two-dimensional subspace in \mathbb{R}^3 , in other words, a plane through the origin. Give two unit vectors which are orthogonal to each other, and which also span the plane.

Solution: Let one of the unit vectors be

$$\hat{u}_1 = \frac{1}{\|v_1\|}v_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

To find the second vector in the basis, let

$$u = v_1 + cv_2,$$

where c is determined so that

$$u \cdot v_1 = 0.$$

Thus,

$$\|v_1\|^2 + c(v_2 \cdot v_1) = 0,$$

which yields that $c = 2$. It then follows that

$$u = v_1 + 2v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

is orthogonal to v_1 . We then let

$$\hat{u}_2 = \frac{1}{\|u\|}u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Observe that \hat{u}_1 and \hat{u}_2 are linearly independent since they are orthogonal and non-zero. To see why this is the case, suppose that

$$c_1\hat{u}_1 + c_2\hat{u}_2 = 0.$$

Taking the inner product with \hat{u}_1 , we get that $c_1 = 0$; taking the inner product with \hat{u}_2 , we get that $c_2 = 0$. Hence the set $\{\hat{u}_1, \hat{u}_2\}$ is linearly independent.

Finally, since both \hat{u}_1 and \hat{u}_2 are in the span of v_1 and v_2 , it follows that $\{\hat{u}_1, \hat{u}_2\}$ also spans $\{v_1, v_2\}$. \square

2. Use an appropriate orthogonal projection to compute the shortest distance from the point $P(1, 1, 2)$ to the plane in \mathbb{R}^3 whose equation is

$$2x + 3y - z = 6.$$

Solution: The point $P_o(0, 0, -6)$ is a point on the plane. Let

$$w = \overrightarrow{P_oP} = \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix};$$

that is, w is the vector from P_o to P . The distance from P to the plane is the norm of the orthogonal projection of w onto the normal direction to the plane $n = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$; namely

$$\|P_n(w)\| = \frac{|w \cdot n|}{\|n\|} = \frac{2}{\sqrt{14}} = \frac{1}{7}\sqrt{14}.$$

□

3. The dual space of \mathbb{R}^n , denoted $(\mathbb{R}^n)^*$, is the vector space of all linear transformations from \mathbb{R}^n to \mathbb{R} .

For a given $w \in \mathbb{R}^n$, define $T_w: \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$T_w(v) = w \cdot v \quad \text{for all } v \in \mathbb{R}^n.$$

Show that T_w is an element of the dual of \mathbb{R}^n for all $w \in \mathbb{R}^n$.

Solution: We need to show that T_w is linear; but this follows from the bi-linearity of the dot product. □

4. Prove that for every linear transformation, $T: \mathbb{R}^n \rightarrow \mathbb{R}$, there exists $w \in \mathbb{R}^n$ such that

$$T(v) = w \cdot v \quad \text{for every } v \in \mathbb{R}^n.$$

(*Hint:* See where T takes the standard basis $\{e_1, e_2, \dots, e_n\}$ in \mathbb{R}^n .)

Solution: For any vector $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, we can write

$$v = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.$$

Thus, since T is linear,

$$\begin{aligned} T(v) &= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) \\ &= [T(e_1) \ T(e_2) \ \cdots \ T(e_n)] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= w^T v, \end{aligned}$$

where w^T denotes the transpose of

$$w = \begin{pmatrix} T(e_1) \\ T(e_2) \\ \vdots \\ T(e_n) \end{pmatrix}.$$

It then follows that $T(v) = w \cdot v$ for all $v \in \mathbb{R}^n$. □

5. Let u_1, u_2, \dots, u_n be unit vectors in \mathbb{R}^n which are mutually orthogonal; that is,

$$u_i \cdot u_j = 0 \quad \text{for } i \neq j.$$

Prove that the set $\{u_1, u_2, \dots, u_n\}$ is a basis for \mathbb{R}^n , and that, for any $v \in \mathbb{R}^n$,

$$v = \sum_{i=1}^n (v \cdot u_i) u_i.$$

Solution: It suffices to prove that the set is linearly independent. Thus, consider the equation

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = \mathbf{0}.$$

Taking the dot product with u_1 on both sides we get

$$c_1 u_1 \cdot u_1 + c_2 u_1 \cdot u_2 + \cdots + c_n u_1 \cdot u_n = 0,$$

which implies that $c_1 = 0$, since the u_i 's are mutually orthogonal. Similarly, we can show that

$$c_2 = c_3 = \cdots = c_n = 0.$$

It then follows that $\{u_1, u_2, \dots, u_n\}$ is linearly independent.

Next, since $\{u_1, u_2, \dots, u_n\}$ is a basis for \mathbb{R}^n , given any vector v in \mathbb{R}^n , there exist scalars c_1, c_2, \dots, c_n such that

$$v = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = \sum_{i=1}^n c_i u_i.$$

Taking the dot product with u_1 on both sides we get that

$$v \cdot u_1 = c_1,$$

since $u_j \cdot u_1 = 0$ when $j \neq 1$ and $u_1 \cdot u_1 = \|u_1\|^2 = 1$. Similarly,

$$c_j = v \cdot u_j \quad \text{for all } i = 1, 2, \dots, n.$$

Consequently,

$$v = \sum_{i=1}^n (v \cdot u_i) u_i.$$

□