Solutions to Assignment #4

1. *(Exercise 19 on page 51 in the text).* Given that $\overrightarrow{r} \cdot \overrightarrow{s} = 0$, $\overrightarrow{r} \cdot \overrightarrow{x} = c$ and $\overrightarrow{r} \times \overrightarrow{x} = \overrightarrow{s}$, find the components of $\overrightarrow{x}$ in each of the three mutually orthogonal directions: $\overrightarrow{r}$, $\overrightarrow{s}$ and $\overrightarrow{r} \times \overrightarrow{s}$.

   ***Solution**:* The component of $\overrightarrow{x}$ along any vector $\overrightarrow{v}$ is given by

   $$P_{\overrightarrow{v}}(\overrightarrow{x}) = \frac{\overrightarrow{x} \cdot \overrightarrow{v}}{\|\overrightarrow{v}\|^2}\overrightarrow{v}.$$

   Accordingly, we have

   $$P_{\overrightarrow{r}}(\overrightarrow{x}) = \frac{\overrightarrow{x} \cdot \overrightarrow{r}}{\|\overrightarrow{r}\|^2}\overrightarrow{r} = \frac{c}{\|\overrightarrow{r}\|^2}\overrightarrow{r},$$

   $$P_{\overrightarrow{s}}(\overrightarrow{x}) = \frac{\overrightarrow{x} \cdot \overrightarrow{s}}{\|\overrightarrow{s}\|^2}\overrightarrow{s} = \overrightarrow{0}$$

   since $\overrightarrow{r} \times \overrightarrow{x} = \overrightarrow{s}$ implies that $\overrightarrow{x}$ is orthogonal to $\overrightarrow{s}$, and

   $$P_{\overrightarrow{x} \times \overrightarrow{s}}(\overrightarrow{x}) = \frac{\overrightarrow{x} \cdot (\overrightarrow{r} \times \overrightarrow{s})}{\|\overrightarrow{r} \times \overrightarrow{s}\|^2}\overrightarrow{r} \times \overrightarrow{s},$$

   where $\overrightarrow{x} \cdot (\overrightarrow{r} \times \overrightarrow{s}) = -\overrightarrow{s} \cdot (\overrightarrow{r} \times \overrightarrow{x}) = -\overrightarrow{s} \cdot \overrightarrow{s} = -\|\overrightarrow{s}\|^2$.
   It then follows that

   $$P_{\overrightarrow{x} \times \overrightarrow{s}}(\overrightarrow{x}) = -\frac{\|\overrightarrow{s}\|^2}{\|\overrightarrow{r} \times \overrightarrow{s}\|^2}\overrightarrow{r} \times \overrightarrow{s}.$$

   □

2. *(Exercise 20 on page 51 in the text).* Prove that the cross product is non–associative; that is, find three vectors $\overrightarrow{r}$, $\overrightarrow{s}$ and $\overrightarrow{t}$ such that

   $$(\overrightarrow{r} \times \overrightarrow{s}) \times \overrightarrow{t} \neq \overrightarrow{r} \times (\overrightarrow{s} \times \overrightarrow{t}).$$

Solution: Take

$$\overrightarrow{r} = \widehat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \overrightarrow{t} = \overrightarrow{s} = \widehat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then,

$$(\overrightarrow{r} \times \overrightarrow{s}) \times \overrightarrow{t} = \widehat{k} \times \widehat{j} = -\widehat{i}$$

while

$$\overrightarrow{r} \times (\overrightarrow{s} \times \overrightarrow{t}) = \widehat{i} \times (\widehat{j} \times \widehat{j}) = \overrightarrow{0}.$$

□

3. *(Exercises 22 and 23 on page 51 in the text).*

   22. Prove that if $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$ and $\overrightarrow{a} \cdot \overrightarrow{b} = 0$, then at least one of $\overrightarrow{a}$ or $\overrightarrow{b}$ must be the zero vector.

       Solution: If none of the two vectors is zero, from $\overrightarrow{a} \cdot \overrightarrow{b} = 0$ we get that the vectors are perpendicular. Thus, the parallelogram determined by the two vectors is actually a square with area

       $$\|\overrightarrow{a} \times \overrightarrow{b}\| = \|\overrightarrow{a}\|\|\overrightarrow{b}\|.$$

       But, $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$ and so

       $$\|\overrightarrow{a}\|\|\overrightarrow{b}\| = 0,$$

       which is impossible unless one of the two vectors is zero. □

   23. Prove that

       $$\overrightarrow{a} \cdot (\overrightarrow{a} \times \overrightarrow{b}) = 0$$

       for any two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.

       Solution: $\overrightarrow{a} \times \overrightarrow{b}$ is orthogonal to $\overrightarrow{a}$; therefore, its dot product with $\overrightarrow{a}$ must be 0. □

4. In this problem and the next, we derive the vector identity

   $$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$$

   for any vectors u, v and w in $\mathbb{R}^3$.

(a) Argue that $u \times (v \times w)$ lies in the span of v and w. Consequently, there exist scalars t and s such that

$$u \times (v \times w) = tv + sw$$

Solution: The vector $u \times (v \times w)$ is orthogonal to the vector $v \times w$; therefore, since $v \times w$ is perpendicular to the plane through spanned by v and w in $\mathbb{R}^3$, it must be the case that $u \times (v \times w)$ also lies in that plane. It then follows that

$$u \times (v \times w) = tv + sw$$

for some scalars t and s. □

(b) Show that $(u \cdot v)t + (u \cdot w)s = 0$.

Solution: Since u is orthogonal to $u \times (v \times w)$, taking inner–product with u on both sides of the equation

$$u \times (v \times w) = tv + sw$$

yields that

$$u \cdot (tv) + u \cdot (sw) = 0$$

or

$$t(u \cdot v) + s(u \cdot w) = 0, \tag{1}$$

which is what we were asked to show. □

5. Let u, v and w be as in the previous problem.

(a) Use the results of the previous problem to conclude that there exists a scalar r such that

$$u \times (v \times w) = r[(u \cdot w)v - (u \cdot v)w].$$

Solution: Assume that none of the vectors u, v and w is zero; otherwise, the identity would hold trivially.
If u is orthogonal to both v and w, then $u \cdot v = u \cdot w = 0$ and so the right–hand side of the identity would be zero. Ont the other hand, u would also be parallel to $v \times w$ and therefore the left–hand side of the identity would also be zero. Thus, in this case, the identity also holds trivially for any scalar r.

Thus, we may assume that $u \cdot v \neq 0$ and $u \cdot w \neq 0$. Multiply

$$u \times (v \times w) = tv + sw$$

by $u \cdot v$ to get

$$(u \cdot v) u \times (v \times w) = t(u \cdot v)v + s(u \cdot v)w.$$

Solving equation (1) for $t(u \cdot v)$ and substituting into the previous equation yields

$$(u \cdot v) u \times (v \times w) = -s(u \cdot w)v + s(u \cdot v)w,$$

which can be re–written as

$$u \times (v \times w) = -\frac{s}{(u \cdot v)}[(u \cdot w)v - (u \cdot v)w].$$

Thus, the identity follows with

$$r = -\frac{s}{(u \cdot v)}.$$

□

(b) By considering some simple examples, deduce that $r = 1$ in the previous identity

***Solution**:* Apply the identity to $u = v = \widehat{i}$ and $w = \widehat{j}$. Then,

$$u \times (v \times w) = \widehat{i} \times (\widehat{i} \times \widehat{j}) = \widehat{i} \times \widehat{k} = -\widehat{j}.$$

On the other hand

$$r[(u \cdot w)v - (u \cdot v)w] = r[(\widehat{i} \cdot \widehat{j})\widehat{i} - (\widehat{i} \cdot \widehat{i})\widehat{j}] = -r\widehat{j}.$$

Thus, $r = 1$. □