Solutions to Assignment #5

1. Let U_1 and U_2 denote subsets in $\mathbb{R}^n$.

   (a) Show that if U_1 and U_2 are open subsets of $\mathbb{R}^n$, then their intersection

   $$U_1 \cap U_2 = \{y \in \mathbb{R}^n \mid y \in U_1 \text{ and } y \in U_2\}$$

   is also open.

   Proof. If the intersection of U_1 and U_2 is empty, then it is open by definition. So suppose that $U_1 \cap U_2 \neq \emptyset$ and let x be any element in $U_1 \cap U_2$. Then, $x \in U_1$ and $x \in U_2$ and so there exist r_1 and r_2, both positive, such that

   $$B_{r_1}(x) \subseteq U_1 \quad \text{and} \quad B_{r_2}(x) \subseteq U_2.$$

   Let r denote the smallest of r_1 and r_2. Then it follows that

   $$B_r(x) \subseteq U_1 \quad \text{and} \quad B_r(x) \subseteq U_2,$$

   which implies that

   $$B_r(x) \subseteq U_1 \cap U_2,$$

   and this shows that $U_1 \cap U_2$ is open. □

   (b) Show that the set $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, y = 0 \right\}$ is not an open subset of $\mathbb{R}^2$.

   Solution: Take, for instance, the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. It lies in the set, but any disc around it contains elements in $\mathbb{R}^2$ that are not in the set. So the set cannot be open. □

2. In Problem 4 of Assignment #2 you proved that every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}$ must be of the form

   $$T(v) = w \cdot v \quad \text{for every } \ v \in \mathbb{R}^n.$$

   Use this fact together with the Cauchy–Schwarz inequality to prove that T is continuous at every point in $\mathbb{R}^n$.

Proof. For any x and y in $\mathbb{R}^n$,

$$T(y) - T(x) = T(y - x) = w \cdot (y - x).$$

It then follows by the Cauchy–Schwarz inequality that

$$\|T(y) - T(x)\| \leqslant \|w\|\|y - x\|.$$

Thus, by the Squeeze Theorem (or the Sandwich Theorem),

$$\lim_{\|y-x\|\to 0} \|T(y) - T(x)\| = 0,$$

which shows that T is continuous at x. Since this is true for any $x \in \mathbb{R}^n$, T is continuous at every point in $\mathbb{R}^n$. □

3. A subset, U, of $\mathbb{R}^n$ is said to be **convex** if given any two points x and y in U, the straight line segment connecting them is entirely contained in U; in symbols,

$$\{x + t(y - x) \in \mathbb{R}^n \mid 0 \leq t \leqslant 1\} \subseteq U$$

(a) Prove that the ball $B_r(O) = \{x \in \mathbb{R}^n \mid \|x\| < R\}$ is a convex subset of $\mathbb{R}^n$.

Proof. Let x and y be in $B_R(O)$; then, $\|x\| < R$ and $\|y\| < R$. For $0 \leqslant t \leqslant 1$, consider

$$x + t(y - x) = (1 - t)x + ty.$$

Thus, taking the norm and using the triangle inequality

$$\begin{aligned} \|x + t(y - x)\| &= \|(1 - t)x + ty\| \\ &\leqslant (1 - t)\|x\| + t\|y\| \\ &< (1 - t)R + tR = R. \end{aligned}$$

Thus, $x + t(y - x) \in B_R(O)$. Since this is true for any $t \in [0, 1]$, it follows that $B_R(O)$ is convex. □

(b) Prove that the "punctured unit disc" in $\mathbb{R}^2$,

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; 0 < x^2 + y^2 < 1 \right\},$$

is not a convex set.

Solution: Take the antipodal points $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ and $\begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}$. The line segment connecting them contains the origin which is not an element of the set. Thus, the set is not convex. □

4. Let x and y denote real numbers.

(a) Starting with the self–evident inequality: $(|x| - |y|)^2 \geqslant 0$, derive the inequality

$$|xy| \leqslant \frac{1}{2}(x^2 + y^2).$$

Proof. Expanding the left–hand side of $(|x| - |y|)^2 \geqslant 0$, we get that

$$x^2 - 2|xy| + y^2 \geqslant 0,$$

from which the inequality follows. □

(b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

Use the inequality derived in the previous part to prove that f is continuous at the origin.

Solution: We show that

$$\lim_{\|(x,y)\| \to 0} |f(x,y) - f(0,0)| = 0.$$

Since, $f(0,0) = 0$, this is equivalent to showing that

$$\lim_{\|(x,y)\| \to 0} |f(x,y)| = 0.$$

Now, by the previous inequality, we have that, for $(x,y) \neq (0,0)$,

$$|f(x,y)| = \frac{|xy|}{\sqrt{x^2+y^2}} \leqslant \frac{1}{2}\frac{x^2+y^2}{\sqrt{x^2+y^2}} = \frac{1}{2}\sqrt{x^2+y^2}.$$

It then follows that

$$0 \leqslant |f(x,y)| \leqslant \frac{1}{2}\|(x,y)\|,$$

and therefore, by the Sandwich Theorem,

$$\lim_{\|(x,y)\|\to 0} |f(x,y)| = 0.$$

□

5. *(Exercise 10 on page 180 in the text).*

   Let

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}, \quad (x,y) \neq (0,0).$$

   Define $f(0,0)$ so that $f(x,y)$ is continuous at $(0,0)$.

   Solution: We define $f(0,0) = 1$. The reason for this is that

$$\lim_{\|(x,y)\|\to 0} f(x,y) = 1.$$

   To see why this is the case, let $u = \|(x,y)\|^2 = x^2 + y^2$; then $u \to 0$ as $\|(x,y)\| \to 0$. Consequently,

$$\lim_{\|(x,y)\|\to 0} f(x,y) = \lim_{u\to 0} \frac{\sin(u)}{u} = 1.$$

□