

Solutions to Assignment #6

1. Use the triangle inequality to prove that, for any x and y in \mathbb{R}^n ,

$$|\|y\| - \|x\|| \leq \|y - x\|.$$

Use this inequality to deduce that the function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$f(x) = \|x\| \quad \text{for all } x \in \mathbb{R}^n$$

is continuous on \mathbb{R}^n .

Proof. Use the triangle inequality to obtain

$$\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\|$$

and

$$\|y\| = \|(y - x) + x\| \leq \|y - x\| + \|x\|.$$

We then get that

$$\|x\| - \|y\| = \|(x - y) + y\| \leq \|x - y\|$$

and

$$-(\|x\| - \|y\|) = \|(y - x) + x\| \leq \|x - y\|.$$

Combining these two expressions yields the result.

From

$$0 \leq |\|y\| - \|x\|| \leq \|y - x\|,$$

for all $x, y \in \mathbb{R}^n$, and the Squeeze Theorem, we obtain that

$$\lim_{\|y-x\| \rightarrow 0} |\|y\| - \|x\|| = 0,$$

from which it follows that $f(x) = \|x\|$ for all $x \in \mathbb{R}^n$ is continuous on all of \mathbb{R}^n . \square

2. Let $f(x, y)$ and $g(x, y)$ denote two functions defined on a open region, D , in \mathbb{R}^2 . Prove that the vector field $F: D \rightarrow \mathbb{R}^2$, defined by

$$F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

is continuous on D if and only if f and g are both continuous on D .

Proof. Let $\begin{pmatrix} x_o \\ y_o \end{pmatrix}$ denote an arbitrary vector in D . We consider

$$F \begin{pmatrix} x \\ y \end{pmatrix} - F \begin{pmatrix} x_o \\ y_o \end{pmatrix} = \begin{pmatrix} f(x, y) - f(x_o, y_o) \\ g(x, y) - g(x_o, y_o) \end{pmatrix}$$

Taking the square of the norm on both sides we get that

$$\left\| F \begin{pmatrix} x \\ y \end{pmatrix} - F \begin{pmatrix} x_o \\ y_o \end{pmatrix} \right\|^2 = (f(x, y) - f(x_o, y_o))^2 + (g(x, y) - g(x_o, y_o))^2.$$

It then follows that

$$\lim_{(x,y) \rightarrow (x_o, y_o)} \left\| F \begin{pmatrix} x \\ y \end{pmatrix} - F \begin{pmatrix} x_o \\ y_o \end{pmatrix} \right\| = 0$$

if and only if

$$\lim_{(x,y) \rightarrow (x_o, y_o)} |f(x, y) - f(x_o, y_o)| = 0 \quad \text{and} \quad \lim_{(x,y) \rightarrow (x_o, y_o)} |g(x, y) - g(x_o, y_o)| = 0.$$

In other words, F is continuous at $\begin{pmatrix} x_o \\ y_o \end{pmatrix}$ if and only if f and g are both continuous at $\begin{pmatrix} x_o \\ y_o \end{pmatrix}$. Since $\begin{pmatrix} x_o \\ y_o \end{pmatrix}$ is arbitrary, the result follows. \square

3. Let U denote an open subset of \mathbb{R}^n and let $F: U \rightarrow \mathbb{R}^m$ and $G: U \rightarrow \mathbb{R}^m$ be two given functions.

(a) Explain how the sum $F + G$ is defined.

Solution: The map $F + G: U \rightarrow \mathbb{R}^m$ is defined as follows

$$(F + G)(x) = F(x) + G(x) \quad \text{for all } x \in U,$$

where the addition on the right-hand side is vector addition in \mathbb{R}^m . \square

(b) Prove that if both F and G are continuous on U , then their sum is also continuous.

(*Suggestion:* The triangle inequality might come in handy.)

Proof. We prove that $F + G$ is continuous at any $x \in U$. Let $y \in U$ and consider

$$\begin{aligned} \|(F + G)(y) - (F + G)(x)\| &= \|F(y) + G(y) - (F(x) + G(x))\| \\ &= \|(F(y) - F(x)) + (G(y) - G(x))\| \\ &\leq \|F(y) - F(x)\| + \|G(y) - G(x)\|, \end{aligned}$$

where we have used the triangle inequality in the last step. It then follows by the Squeeze (or Sandwich) Theorem that, if F and G are both continuous at x , then

$$\lim_{\|y-x\| \rightarrow 0} \|(F + G)(y) - (F + G)(x)\| = 0.$$

Thus, $F + G$ is continuous at x . Since, x is an arbitrary element of U , it also follows that $F + G$ is continuous on U . \square

4. In each of the following, given the function $F: U \rightarrow \mathbb{R}^m$ and the set B , compute the pre-image $F^{-1}(B)$.

(a) $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$, and $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

Solution:

$$\begin{aligned} F^{-1} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \text{ and } x^2 - y^2 = 0 \right\} \\ &= \left\{ \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}, \begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix}, \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}, \begin{pmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix} \right\}. \end{aligned}$$

\square

(b) $f: D' \rightarrow \mathbb{R}$,

$$f(x, y) = \frac{1}{\sqrt{1 - x^2 - y^2}}, \quad \text{for } (x, y) \in D'$$

where $D' = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1\}$ (the punctured unit disc), $B = \{1\}$.

Solution:

$$\begin{aligned} f^{-1}(1) &= \{(x, y) \in D' \mid f(x, y) = 1\} \\ &= \left\{ (x, y) \in D' \mid \frac{1}{\sqrt{1 - x^2 - y^2}} = 1 \right\}. \end{aligned}$$

Since the equation

$$\frac{1}{\sqrt{1-x^2-y^2}} = 1$$

only has $(0,0)$ as a solution, and $(0,0)$ is not in D' , it follows that $f^{-1}(1) = \emptyset$, the empty set. \square

(c) $f: D' \rightarrow \mathbb{R}$ is as in part (b), and $B = \{2\}$.

Solution: To compute $f^{-1}(2)$, solve the equation

$$\frac{1}{\sqrt{1-x^2-y^2}} = 2$$

or

$$1 - x^2 - y^2 = \frac{1}{4}$$

or the equation of the circle

$$x^2 + y^2 = \frac{3}{4}$$

of radius $\sqrt{3}/2$ around the origin in \mathbb{R}^2 . Thus,

$$f^{-1}(2) = \left\{ (x, y) \in D \mid x^2 + y^2 = \frac{3}{4} \right\}.$$

\square

(d) $f: D' \rightarrow \mathbb{R}$ is as in part (b), and $B = \{1/2\}$.

Solution: To compute $f^{-1}(1/2)$, solve the equation

$$\frac{1}{\sqrt{1-x^2-y^2}} = \frac{1}{2},$$

or

$$1 - x^2 - y^2 = 4,$$

or

$$x^2 + y^2 = -3.$$

This equation has not real solutions. Therefore, $f^{-1}(1/2) = \emptyset$, the empty set. \square

5. Compute the image the given sets under the following maps

- (a) $\sigma: \mathbb{R} \rightarrow \mathbb{R}^2$, $\sigma(t) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$. Compute $\sigma(\mathbb{R})$.

Solution:

$$\sigma(\mathbb{R}) = \{(x, y) \in \mathbb{R}^2 \mid x = \cos t, y = \sin t, t \in \mathbb{R}\},$$

which is the same as the set

$$\sigma(\mathbb{R}) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

or the unit circle in \mathbb{R}^2 . □

- (b) $f: D' \rightarrow \mathbb{R}$ and D' are as given in part (b) of the previous problem. Compute $f(D')$.

Solution:

$$f(D') = \{t \in \mathbb{R} \mid t = f(x, y) \text{ for some } (x, y) \in D'\}.$$

Now, if

$$t = \frac{1}{\sqrt{1 - x^2 - y^2}}, \text{ where } 0 < x^2 + y^2 < 1,$$

it follows that $1 < t < \infty$, and therefore

$$f(D') = (1, \infty),$$

the open interval from 0 to ∞ . □