Solutions to Assignment #7

1. Let U denote an open subset of $\mathbb{R}^n$. Suppose that $f\colon U \to \mathbb{R}$ is a scalar field and $G\colon U \to \mathbb{R}^m$ is vector valued function.

   (a) Explain how the product fG is defined.

   Solution: The product of f and G is the function $fG\colon U \to \mathbb{R}^n$ defined by

   $$(fG)(x) = f(x)G(x) \quad \text{for all} \quad x \in U;$$

   i.e., scalar multiplication of the vector $G(x)$ by the scalar $f(x)$. $\square$

   (b) Prove that if both f and G are continuous on U, then the vector valued function fG is also continuous on U.

   Proof: For any u and v in U, observe that

   $$\begin{aligned} \|(fG)(v) - (fG)(u)\| &= \|f(v)G(v) - f(u)G(u)\| \\ &= \|f(v)G(v) - f(v)G(u) + f(v)G(u) - f(u)G(u)\| \\ &= \|f(v)(G(v) - G(u)) + (f(v) - f(u))G(u)\| \end{aligned}$$

   Thus, by the triangle inequality,

   $$\|(fG)(v) - (fG)(u)\| \leqslant \|f(v)(G(v) - G(u))\| + \|(f(v) - f(u))G(u)\|,$$

   which yields

   $$\|(fG)(v) - (fG)(u)\| \leqslant |f(v)|\|G(v) - G(u)\| + |f(v) - f(u)|\|G(u)\|.$$

   The fact that

   $$\lim_{\|v-u\|\to 0} \|(fG)(v) - (fG)(u)\| = 0$$

   now follows from this inequality, the Squeeze Theorem, and the assumptions that f and G are continuous. $\square$

2. Let U be an open subset of $\mathbb{R}^2$. Let $f\colon U \to \mathbb{R}$ and $g\colon U \to \mathbb{R}$ be two scalar fields on U, and define $h\colon U \to \mathbb{R}$ by

   $$h(x, y) = f(x, y)g(x, y) \quad \text{for all} \quad (x, y) \in U.$$

   Prove that if both f and g are continuous on U, then so is h.

Suggestion: First prove that the function $G\colon \mathbb{R}^2 \to \mathbb{R}$, defined by $G(x,y) = xy$ for all $(x,y) \in \mathbb{R}^2$, is continuous. Then, let $F\colon U \to \mathbb{R}^2$ denote the map given by

$$F(x,y) = (f(x,y), g(x,y)) \quad \text{for all} \quad (x,y) \in U,$$

and observe that

$$h = G \circ F.$$

Solution: We first prove that the function $G\colon \mathbb{R}^2 \to \mathbb{R}$, defined by $G(x,y) = xy$ for all $(x,y) \in \mathbb{R}^2$, is continuous.
Using the triangle inequality we obtain

$$\begin{aligned} |G(x,y) - G(x_o,y_o)| &= |xy - x_o y_o| \\ &= |xy - x_o y + x_o y - x_o y_o| \\ &= |(x - x_o)y + x_o(y - y_o)| \\ &\leqslant |x - x_o|\ |y| + |x_o|\ |y - y_o|. \end{aligned}$$

Next, use the estimates

$$|x - x_o| \leqslant \|(x,y) - (x_o,y_o)\|$$

and

$$|y - y_o| \leqslant \|(x,y) - (x_o,y_o)\|$$

to obtain

$$|G(x,y) - G(x_o,y_o)| \leqslant \|(x,y) - (x_o,y_o)\|\ |y| + |x_o|\ \|(x,y) - (x_o,y_o)\|,$$

or

$$|G(x,y) - G(x_o,y_o)| \leqslant (|y| + |x_o|)\ \|(x,y) - (x_o,y_o)\|.$$

Observe that

$$\lim_{(x,y)\to(x_o,y_o)} |y| = |y_o|,$$

which follows from the fact that the map $(x,y) \to y$ is continuous since it is a projection. Thus,

$$\lim_{(x,y)\to(x_o,y_o)} (|y| + |x_o|)\ \|(x,y) - (x_o,y_o)\| = (|y_o| + |x_o|) \cdot 0 = 0,$$

Hence, from

$$0 \leqslant |G(x,y) - G(x_o,y_o)| \leqslant (|y| + |x_o|)\ \|(x,y) - (x_o,y_o)\|,$$

and the Sandwich theorem, it follows that

$$\lim_{(x,y)\to(x_o,y_o)} |G(x,y) - G(x_o,y_o)| = 0,$$

which shows that G is continuous.
Next, observe that since f and g are continuous, the map $F\colon U \to \mathbb{R}^2$ given by

$$F(x,y) = (f(x,y), g(x,y)) \quad \text{for all} \quad (x,y) \in U,$$

is also continuous.
Note also that

$$\begin{aligned} h(x,y) &= f(x,y)g(x,y) \\ &= G(f(x,y), g(x,y)) \\ &= G(F(x,y)) \\ &= G \circ F(x,y); \end{aligned}$$

that is,

$$h = G \circ F$$

and therefore h is continuous, since it is the composition of two continuous functions. □

3. Let $U = \mathbb{R}^n \backslash \{\mathbf{0}\} = \{v \in \mathbb{R}^n \mid v \neq \mathbf{0}\}$.

(a) Prove that U is an open subset of $\mathbb{R}^n$.

Proof: Let $u \in U$. Then, $\|u\| > 0$. Consider the ball $B_r(u)$, where $r = \dfrac{\|u\|}{2}$. Then, $v \in B_r(u)$ implies that

$$\|u\| = \|u - v + v\| \leqslant \|v - u\| + \|v\| < r + \|v\| = \frac{\|u\|}{2} + \|v\|.$$

It then follows that

$$\|v\| > \|u\| - \frac{\|u\|}{2} = \frac{\|u\|}{2} > 0,$$

which shows that $v \in U$. Hence,

$$B_r(u) \subseteq U.$$

Since $u \in U$ was arbitrary, it follows that U is open. □

(b) Define $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(v) = \frac{1}{\|v\|} \quad \text{for all} \quad v \in U.$$

Prove that f is continuous on U.
Suggestion: Note that the function, g, defined by

$$g(t) = \frac{1}{t} \quad \text{for all } t \neq 0,$$

is continuous for $t \neq 0$.

Proof: Let $g(t) = \dfrac{1}{t}$ for $t \neq 0$. Then, g is co, continuous for $t \neq 0$. The function $h\colon U \to \mathbb{R}$, where U is defined above, defined by

$$h(v) = \|v\| \quad \text{for all } v \in U$$

is continuous as proves in Problem 1 in Assignment #6. Since, $h(v) \neq 0$ for all $v \in U$, it follows that the composition

$$g \circ h(v) = g(h(v)) = \frac{1}{\|v\|}$$

is defined for $v \in U$. Since $f = g \circ h$ and both g and h are continuous on U, it follows that f is continuous on U. □

4. Let $I \subseteq \mathbb{R}$ be an open interval and $\sigma\colon I \to \mathbb{R}^n$ be continuous path in $\mathbb{R}^n$ satisfying $\sigma(t) \neq \mathbf{0}$ for all $t \in I$. Define the function $f\colon I \to \mathbb{R}$ by

$$f(t) = \frac{1}{\|\sigma(t)\|} \quad \text{for all} \quad t \in I.$$

Prove that f is continuous on I.

Proof: Let U be defined as in Problem 3 and let $g(v) = \dfrac{1}{\|v\|}$ for all $v \in U$. Then g is continuous on U by Problem 3. Furthermore $\sigma(I) \subseteq U$ by the assumption that $\|\sigma(t)\| \neq 0$ for all $t \in I$. It then follows that the composition $g \circ \sigma \colon I \to \mathbb{R}$ is defined and

$$f(t) = g \circ \sigma(t) \quad \text{for all} \quad t \in I.$$

Since both g and σ are continuous, it follows that f is continuous on I. □

5. *(Exercise 12 on page 180 in the text).*

   Let

$$f(x,y) = \frac{x-y}{x+y}, \quad x+y \neq 0.$$

   Can this function be defined on the line $x+y=0$ so that it is continuous at some point on this line?

   Solution*:* Let x_o denote a non–zero real number, and consider the point $(x_o, -x_o)$ on the line $x+y=0$. The path $\sigma(t) = (x_o+t, -x_o+t)$ is continuous and has the property that $\sigma(t) \to (x_o, -x_o)$ as $t \to 0$. Note that

$$f(\sigma(t)) = \frac{x_o}{t} \quad \text{for} \quad t \neq 0.$$

   Thus, unless $x_o = 0$, the limit

$$\lim_{t\to 0} f(\sigma(t))$$

   does not exist.

   On the other hand,

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

   does not exist either since approaching along the paths $\sigma_1(t) = (t,0)$ and $\sigma_2(t) = (0,t)$ as $t \to 0$ yield different values.

   Consequently, at not point on the line $x+y=0$ can f be defined so that it is continuous. □