Solutions to Assignment #8

1. Let f denote a real valued function defined on some open interval around $a \in \mathbb{R}$. Consider a line of slope m and equation

$$L(x) = f(a) + m(x - a) \quad \text{for all} \quad x \in \mathbb{R}.$$

Suppose that this line if the best approximation to the function f at a in the sense that

$$\lim_{x \to a} \frac{|E(x)|}{|x - a|} = 0,$$

where $E(x) = f(x) - L(x)$ for all x in the interval in which f is defined.

Prove that f is differentiable at a, and that $f'(a) = m$.

***Solution**:* Observe that

$$E(x) = f(x) - f(a) - m(x - a),$$

so that

$$\frac{|E(x)|}{|x - a|} = \left| \frac{f(x) - f(a)}{x - a} - m \right|.$$

It then follows that

$$\lim_{x \to a} \left| \frac{f(x) - f(a)}{x - a} - m \right| = 0$$

and therefore

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = m.$$

This shows that f is differentiable at a and $f'(a) = m$. □

2. Recall that a function $F\colon U \to \mathbb{R}^m$, where U is an open subset for $\mathbb{R}^n$, is said to be differentiable at $u \in U$ if and only if there exists a unique linear transformation $T_u\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\|v - u\| \to 0} \frac{\|F(v) - F(u) - T_u(y - x)\|}{\|v - u\|} = 0.$$

Prove that if F is differentiable at u, then it is also continuous at u.

Give an example that shows that the converse of this assertion is not true.

Proof. Suppose that F is differentiable at $u \in U$. Then,

$$F(v) - F(u) = T_u(v-u) + E_u(v-u),$$

where

$$E_u(w) = F(u+w) - F(u) - T_w(w)$$

has the property that

$$\lim_{\|w\|\to 0} \frac{\|E_u(w)\|}{\|w\|} = 0. \tag{1}$$

It then follows, by the triangle inequality, that

$$\|F(v) - F(u)\| \leqslant \|T_u(v-u)\| + \|E_u(v-u)\|. \tag{2}$$

Now, since T_u is linear, it is also continuous and therefore

$$\lim_{\|v-u\|\to 0} \|T_u(v-u)\| = 0.$$

Also, it follows from (1) that

$$\lim_{\|v-u\|\to 0} \|E_u(v-u)\| = \lim_{\|v-u\|\to 0} \left\{ \|v-u\| \cdot \frac{\|E_u(v-u)\|}{\|v-u\|} \right\} = 0.$$

Hence, by the Sandwich Theorem, it follows from (2) that

$$\lim_{\|v-u\|\to 0} \|F(v) - F(u)\| = 0,$$

and therefore F is continuous at u. $\square$

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \sqrt{|xy|}$ for all $(x,y) \in \mathbb{R}^2$. Show that f is not differentiable at $(0,0)$.

   Proof: If f was differentiable at $(0,0)$, there would be a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}$ such that

   $$f(h,k) = T(h,k) + E(h,k),$$

   where E has the property that

   $$\lim_{(h,k)\to(0,0)} \frac{|E(h,k)|}{\sqrt{h^2+k^2}} = 0. \tag{3}$$

In particular, if we take $h = k = t$ where $t \in \mathbb{R}$ is sufficiently small, we get that

$$|t| = T(t,t) + E(t,t).$$

Then, since T is linear

$$|t| = tT(1,1) + E(t,t).$$

Dividing by $t \neq 0$, we then get that

$$\frac{|t|}{t} = T(1,1) + \frac{E(t,t)}{t},$$

where

$$\lim_{|t|\to 0} \frac{|E(t,t)|}{|t|} = 0$$

by (3). Hence,

$$\lim_{t\to 0} \frac{|t|}{t} = T(1,1).$$

However, $\lim_{t\to 0} \frac{|t|}{t}$ does not exist. This contradiction yields that $f(x,y)$ cannot be differentiable at $(0,0)$. □

4. *(Exercise 4 on page 197 in the text).*
   Is $f(x,y,z) = x\sqrt{y^2+z^2}$ differentiable at $(0,0,0)$? Prove your assertion.

   ***Solution**:* Yes, f is differentiable at $(0,0,0)$. In fact, the derivative of f at $(0,0,0)$ is the zero transformation; i.e., $T(x,y,z) = 0$ for all $(x,y,z) \in \mathbb{R}^3$. To show this, we prove that

$$\lim_{(x,y,z)\to(0,0,0} \frac{|f(x,y,z) - f(0,0,0) - T(x,y,z)|}{\sqrt{x^2+y^2+z^2}} = 0$$

or

$$\lim_{(x,y,z)\to(0,0,0)} \frac{|x|\sqrt{y^2+z^2}}{\sqrt{x^2+y^2+z^2}} = 0.$$

Now, for $(x,y,z) \neq (0,0,0)$,

$$\sqrt{y^2+z^2} \leqslant \sqrt{x^2+y^2+z^2},$$

from which we get that

$$\frac{\sqrt{y^2+z^2}}{\sqrt{x^2+y^2+z^2}} \leqslant 1.$$

Thus,

$$0 \leqslant \frac{|x|\sqrt{y^2+z^2}}{\sqrt{x^2+y^2+z^2}} \leqslant |x|.$$

Hence, since $|x| \to 0$ as $\sqrt{x^2+y^2+z^2} \to 0$, it follows from the Sandwich Theorem that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{|x|\sqrt{y^2+z^2}}{\sqrt{x^2+y^2+z^2}} = 0.$$

□

5. *(Exercise 6 on page 197 in the text).*

Is the scalar field

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

continuous at the origin? Is it differentiable at the origin?

***Solution**:* Yes, f is continuous at $(0,0)$. To prove this, use the inequality

$$|xy| \leqslant \frac{1}{2}(x^2+y^2)$$

to get that, for $(x,y) \neq (0,0)$,

$$0 \leqslant \frac{|xy|}{\sqrt{x^2+y^2}} \leqslant \frac{1}{2}\sqrt{x^2+y^2}.$$

Thus, by the Sandwich Theorem,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

and therefore f is continuous at $(0,0)$. However, f is not differentiable at the origin. To see why this is the case, assume that there is a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,y) = T(x,y) + E(x,y),$$

where

$$\lim_{(x,y)\to(0,0)} \frac{|E(x,y)|}{\sqrt{x^2+y^2}} = 0.$$

Then, for $t \neq 0$ we have that

$$f(t,t) = T(t,t) + E(t,t),$$

or, using the linearity of T,

$$\frac{|t|}{\sqrt{2}} = tT(1,1) + E(t,t).$$

Dividing by $t \neq 0$,

$$\frac{1}{\sqrt{2}}\frac{|t|}{t} = T(1,1) + \frac{E(t,t)}{t},$$

where

$$\lim_{t\to 0} \frac{|E(t,t)|}{|t|} = \sqrt{2}\lim_{t\to 0} \frac{|E(t,t)|}{\sqrt{t^2+t^2}} = 0.$$

Hence,

$$\lim_{t\to 0} \frac{|t|}{t} = \sqrt{2}\; T(1,1).$$

However, $\lim_{t\to 0} \frac{|t|}{t}$ does not exist. This contradiction yields that $f(x,y)$ cannot be differentiable at $(0,0)$. $\square$