Solutions to Assignment #9

1. *(Exercises 7(a) and 7(b) on pages 197 and 198 in the text).*

   Let
$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right), & \text{if } (x,y)\neq(0,0) \\ 0, & \text{if } (x,y)=(0,0) \end{cases}$$

   (a) Show that the partial derivatives of f with respect to x and y are not continuous at $(0,0)$.

   (b) Show that the partial derivatives of f with respect to x and y do exist at $(0,0)$ and are equal to 0 there. It follows that if f is differentiable at $(0,0)$, then
$$Df(0,0) = (0,0).$$

   Solution: First compute the partial derivatives. If $(x,y)\neq(0,0)$, then
$$\frac{\partial f}{\partial x}(x,y) = 2x\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right)$$
   and
$$\frac{\partial f}{\partial y}(x,y) = 2y\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right).$$
   For $(x,y)=(0,0)$, we compute
$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} \\ &= \lim_{t\to 0} t\sin\left(\frac{1}{t^2}\right) \\ &= 0,\end{aligned}$$
   since
$$\left|\sin\left(\frac{1}{t^2}\right)\right| \leqslant 1 \quad \text{for all} \quad t\neq 0.$$
   Similarly,
$$\frac{\partial f}{\partial y}(0,0) = 0.$$

We then have that

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2x\left(\sin\left(\dfrac{1}{x^2+y^2}\right) - \dfrac{1}{x^2+y^2}\cos\left(\dfrac{1}{x^2+y^2}\right)\right), & \text{if } (x,y)\neq(0,0) \\ 0, & \text{if } (x,y)=(0,0) \end{cases}$$

and

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} 2y\left(\sin\left(\dfrac{1}{x^2+y^2}\right) - \dfrac{1}{x^2+y^2}\cos\left(\dfrac{1}{x^2+y^2}\right)\right), & \text{if } (x,y)\neq(0,0) \\ 0, & \text{if } (x,y)=(0,0) \end{cases}$$

To see that the partial derivatives of f are not continuous at $(0,0)$, observe that

$$\lim_{t\to 0}\frac{\partial f}{\partial x}(t,0) = \lim_{t\to 0}\left\{2t\sin\left(\frac{1}{t^2}\right) - \frac{2}{t}\cos\left(\frac{1}{t^2}\right)\right\}$$

does not exist since

$$\lim_{t\to 0}\frac{2}{t}\cos\left(\frac{1}{t^2}\right)$$

does not exist. Hence, $\dfrac{\partial f}{\partial x}$ is not continuous at $(0,0)$. A similar calculation shows that $\dfrac{\partial f}{\partial y}$ is not continuous at $(0,0)$ either. □

2. *(Exercise 7(c) on page 198 in the text).*

Let f be as in the previous problem. Show that f is differentiable at $(0,0)$.

***Solution**:* We need to show that

$$f(x,y) = f(0,0) + Df(0,0)(x,y) + E(x,y)$$

where $Df(0,0)(x,y) = 0$ for all $(x,y)\in\mathbb{R}^2$; i.e., $Df(0,0)$ is the zero map, and

$$\lim_{(x,y)\to(0,0)}\frac{|E(x,y)|}{\sqrt{x^2+y^2}} = 0. \tag{1}$$

In this case,

$$E(x,y) = (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) \quad \text{for } (x,y)\neq(0,0)$$

so that

$$0 \leqslant \frac{|E(x,y)|}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}\left|\sin\left(\frac{1}{x^2+y^2}\right)\right| \leqslant \sqrt{x^2+y^2}$$

for $(x,y) \neq (0,0)$; thus, (1) follows by the Sandwich Theorem, and therefore f is differentiable at $(0,0)$. □

3. *(Exercise 8 on page 198 in the text).*

   Show that $f(x,y) = \sqrt{|xy|}$ is continuous at $(0,0)$, but $\partial f/\partial x$ and $\partial f/\partial x$ are not continuous at $(0,0)$. Is f differentiable at $(0,0)$?

   ***Solution**:* First observe that

$$|xy| = |x||y| \leqslant \frac{1}{2}(x^2+y^2),$$

so that

$$\sqrt{|xy|} \leqslant \frac{1}{\sqrt{2}}\sqrt{x^2+y^2}.$$

It then follows by the Sandwich Theorem that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0),$$

and therefore f is continuous at $(0,0)$.

The partial derivatives of f at $(0,0)$ exist. To see why this is the case, compute

$$\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = 0,$$

so that $\dfrac{\partial f}{\partial x}(0,0) = 0$. Similarly, $\dfrac{\partial f}{\partial y}(0,0) = 0$. However, the partial derivatives of f do not exist for points $(x,y) \neq (0,0)$ such that $xy = 0$. To see this, let $x_o \neq 0$ and consider, for $t > 0$,

$$\frac{f(x_o,t)-f(0,0)}{t} = \frac{\sqrt{|x_o||t|}}{t} = \frac{\sqrt{|x_o|}}{\sqrt{t}};$$

so that

$$\lim_{t\to 0^+}\frac{f(x_o,t)-f(0,0)}{t} = +\infty,$$

and therefore $\dfrac{\partial f}{\partial y}(x_o, 0)$ does not exist for $x_o \neq 0$. Similarly, $\dfrac{\partial f}{\partial x}(0, y_o)$ does not exist for $y_o \neq 0$.

The partial derivatives of f fail to be continuous at $(0, 0)$ even in the region where they are defined. For instance, consider the first quadrant $Q_+ = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ together with the origin. Here

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} \dfrac{y}{2\sqrt{xy}} & \text{if } (x, y) \in Q_+ \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

It we evaluate the limit of $\partial f/\partial x$ along the path $\sigma(t) = (t, t)$ as $t \to 0^+$, we obtain that

$$\lim_{t\to 0^+} \frac{\partial f}{\partial x}(t, t) = \frac{1}{2} \neq 0.$$

Consequently, $\partial f/\partial x$ cannot be continuous at $(0, 0)$. The same holds true for $\partial f/\partial y$.

f is not differentiable at $(0, 0)$. To see why this is so, suppose that f was differentiable at $(0, 0)$. Then the derivative map $Df(0, 0)$ would have to be the zero map (since the partial derivatives at $(0, 0)$ are 0). Then,

$$f(x, y) = E(x, y),$$

where

$$\lim_{(x,y)\to(0,0)} \frac{|E(x, y)|}{\sqrt{x^2 + y^2}} = 0.$$

However,

$$\lim_{(x,y)\to(0,0)} \frac{|E(x, y)|}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{\sqrt{x^2 + y^2}}$$

does not exist. To see why this is so, observe that

$$\lim_{t\to 0} \frac{|E(t, 0)|}{\sqrt{x^2 + y^2}} = 0,$$

but

$$\lim_{t\to 0} \frac{|E(t, t)|}{\sqrt{x^2 + y^2}} = \frac{1}{\sqrt{2}} \neq 0.$$

Consequently, f cannot be differentiable at $(0, 0)$. □

4. *Exercises 14(b) and 14(c) on pages 198 and 199 in the text.*

   Find the gradient of f for each of the following scalar fields:

   (b) $f(x, y, z) = xe^{yz}$,

   Answer:

$$\nabla f(x, y, z) = e^{yz}\,\widehat{i} + xze^{yz}\,\widehat{j} + xye^{yz}\,\widehat{k}.$$

□

   (c) $f(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$, $(x, y, z) \neq (0, 0, 0)$.

   Answer:

$$\nabla f(x, y, z) = -\frac{1}{(x^2 + y^2 + z^2)^{3/2}}(x\,\widehat{i} + y\,\widehat{j} + z\,\widehat{k}).$$

□

5. Find the mixed partial derivatives

$$\frac{\partial^2 f}{\partial x \partial y},\ \frac{\partial^2 f}{\partial y \partial x},\ \frac{\partial^2 f}{\partial x \partial z},\ \frac{\partial^2 f}{\partial z \partial x},\ \frac{\partial^2 f}{\partial y \partial z},\ \frac{\partial^2 f}{\partial z \partial y},$$

   for the scalar fields given in Exercises 14(b) and 14(c) on pages 198 and 199 in the text.

   (b) $f(x, y, z) = xe^{yz}$,

   Solution: In this case,

$$\frac{\partial f}{\partial x} = e^{yz},\ \frac{\partial f}{\partial y} = xze^{yz} \text{ and } \frac{\partial f}{\partial z} = xye^{yz}.$$

   We then have that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = ze^{yz} \text{ and } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = ze^{yz};$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = ye^{yz} \text{ and } \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = ye^{yz};$$

   and

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = (x + xyz)e^{yz} \text{ and } \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = (x + xyz)e^{yz}.$$

□

(c) $f(x,y,z) = 1/\sqrt{x^2+y^2+z^2}$, $(x,y,z) \neq (0,0,0)$.

Solution: In this case

$$\frac{\partial f}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{3/2}},$$

$$\frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{3/2}},$$

and

$$\frac{\partial f}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{3/2}}.$$

It then follows that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{3xy}{(x^2+y^2+z^2)^{5/2}},$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = \frac{3xz}{(x^2+y^2+z^2)^{5/2}},$$

and

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = \frac{3yz}{(x^2+y^2+z^2)^{5/2}},$$

□