Solutions to Exam 1

1. The points $P(1,0,0)$, $Q(0,2,0)$ and $R(0,0,3)$ determine a unique plane in three dimensional Euclidean space, $\mathbb{R}^3$.

   (a) Give the equation of the plane determined by P, Q and R.

   Solution: First, we find a vector, n, which is perpendicular to the plane by computing

$$\begin{aligned} n &= \overrightarrow{PR} \times \overrightarrow{PQ} \\ &= \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} \\ &= 6\widehat{i} + 3\widehat{j} + 2\widehat{k}. \end{aligned}$$

   It then follows that the equation of the plane determined by P, Q and R is

$$6(x-1) + 3y + 2z = 0,$$

   or

$$6x + 3y + 2z = 6.$$

   □

   (b) Give the parametric equations of the line through the point $(1,1,1)$ which is perpendicular to the plane determined by P, Q and R.

   Answer: The parametric equations of the line through $(1,1,1)$ in the direction of $n = 6\widehat{i} + 3\widehat{j} + 2\widehat{k}$ are

$$\begin{cases} x = 1 + 6t \\ y = 1 + 3t \\ z = 1 + 2t \end{cases}$$

   where $t \in \mathbb{R}$. □

   (c) Find the intersection between the line found in part (b) above and the plane determined by P, Q and R.

Solution: Solve the equation

$$6(1+6t)+3(1+3t)+2(1+2t)=6$$

to obtain that $t=-\dfrac{5}{49}$. Thus, the intersection between the line found in part (b) above and the plane determined by P, Q and R is determined by the coordinates

$$\begin{cases} x = 1-6\cdot\dfrac{5}{49}=\dfrac{19}{49} \\[2ex] y = 1-3\cdot\dfrac{5}{49}=\dfrac{34}{49} \\[2ex] z = 1-2\cdot\dfrac{5}{49}=\dfrac{39}{49} \end{cases}$$

which yield the point $\left(\dfrac{19}{49},\dfrac{34}{49},\dfrac{39}{49}\right)$. □

2. Let P, Q and R be the points given in Problem 1.

   (a) Give the coordinates of the point in the plane determined by P, Q and R which is the closest to the point $(1,1,1)$.

   Answer: The point on the plane determined by P, Q and R which is the closest to the point $(1,1,1)$ lies on the line through this $(1,1,1)$ which is perpendicular to the plane. This point was found in part (c) of Problem 1 to be $\left(\dfrac{19}{49},\dfrac{34}{49},\dfrac{39}{49}\right)$. □

   (b) Find the (shortest) distance from the point $(1,1,1)$ to the plane determined by P, Q and R.

   Answer: Compute the distance between $(1,1,1)$ and $\left(\dfrac{19}{49},\dfrac{34}{49},\dfrac{39}{49}\right)$ to get $\dfrac{5}{7}$. □

3. Let P, Q and R be the points given in Problem 1.

   Give the area of the triangle whose vertices are P, Q and R.

Answer: Compute

$$\begin{aligned}\text{area}(\triangleleft PQR) &= \frac{1}{2}\|\overrightarrow{PR}\times\overrightarrow{PQ}\| \\ &= \frac{1}{2}\|6\widehat{i}+3\widehat{j}+2\widehat{k}\| \\ &= \frac{7}{2}.\end{aligned}$$

□

4. Let U denote an open subset of $\mathbb{R}^n$, and let $F\colon U \to \mathbb{R}^m$ be a vector valued function defined on U.

   (a) State precisely what it means for F to be continuous at $u \in U$.

   Answer: F is continuous at $u \in U$ means that

   $$\lim_{\|v-u\|\to 0}\|F(v)-F(u)\| = 0.$$

   □

   (b) Assume that there is a constant $K \geqslant 0$ such that

   $$\|F(v_1)-F(v_2)\| \leqslant K\|v_1 - v_2\| \quad \text{for all } v_1, v_2 \in U. \tag{1}$$

   Prove that F is continuous on U.

   Proof: Let u denote an arbitrary point in U. Since u is open, there exists $r > 0$ such that

   $$B_r(u) \subseteq U.$$

   For any $v \in B_r(u)$, by the condition in (1),

   $$0 \leqslant \|F(v)-F(u)\| \leqslant K\|v-u\|.$$

   Thus, by the Squeeze Theorem,

   $$\lim_{\|v-u\|\to 0}\|F(v)-F(u)\| = 0,$$

   which shows that F is continuous at u. Since this is the case for arbitrary $u \in U$, we conclude that F is continuous on U. □

5. Given $w \in \mathbb{R}^n$, define $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(v) = w \cdot v \quad \text{for all } v \in \mathbb{R}^n;$$

that is, $f(v)$ is the dot product of w with v.

(a) Use the Cauchy–Schwarz inequality to verify that f satisfies the condition (1) in part (b) of Problem 4; namely,

$$|f(v_1) - f(v_2)| \leqslant K\|v_1 - v_2\| \quad \text{for all } v_1, v_2 \in \mathbb{R}^n.$$

What is K in this case?
Deduce therefore that f is continuous on $\mathbb{R}^n$.

Solution: For any v_1 and v_2 in $\mathbb{R}^n$,

$$f(v_1) - f(v_2) = w \cdot v_1 - w \cdot v_2 = w \cdot (v_1 - v_2).$$

Thus, by the Cauchy-Schwarz inequality,

$$|f(v_1) - f(v_2)| = |w \cdot (v_1 - v_2)| \leqslant \|w\|\|v_1 - v_2\|,$$

which is condition (1) with $K = \|w\|$.
It then follows from the result in part (b) of Problem 4 that f is continuous on $\mathbb{R}^n$. $\square$

(b) Deduce also that the function $P_i(x_1, x_2, \ldots, x_n) = x_i$, for all points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, is continuous on $\mathbb{R}^n$; where x_i denotes the i^{th} coordinate of the point $(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, n$. Explain your reasoning.

Solution: Let $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis in $\mathbb{R}^n$. Then, for each $i = 1, 2, \ldots, n$,

$$P_i(v) = e_i \cdot v \quad \text{for all } v \in \mathbb{R}^n.$$

That P_i is of the form in part (a) of this problem. Consequently, P_i is continuous on $\mathbb{R}^n$. $\square$

6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{|x|y}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0); \\[2ex] 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f is continuous at $(0,0)$.

Proof. For $(x, y) \neq (0, 0)$,

$$|f(x, y)| = \frac{|x||y|}{\sqrt{x^2 + y^2}} \leqslant |y|,$$

since $|x| \leqslant \sqrt{x^2 + y^2}$ for all $(x, y) \in \mathbb{R}^2$. Consequently,

$$0 \leqslant |f(x, y) - f(0, 0)| \leqslant \sqrt{x^2 + y^2} \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

it then follows by the Squeeze Theorem that

$$\lim_{\|(x,y)-(0,0)\| \to 0} |f(x, y) - f(0, 0)| = 0,$$

which shows that f is continuous at $(0, 0)$. □

7. Is the function $f\colon \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x, y) = \begin{cases} \dfrac{|x|}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0); \\ \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

continuous at $(0, 0)$? Justify your answer.

Solution: Let $\varepsilon = \dfrac{1}{2}$ and observe that for any $\delta > 0$

$$f\left(\frac{\delta}{2}, 0\right) = 1.$$

Thus,

$$\left\|\left(\frac{\delta}{2}, 0\right)\right\| = \frac{\delta}{2} < \delta,$$

but

$$\left|f\left(\frac{\delta}{2}, 0\right) - f(0, 0)\right| = 1 > \frac{1}{2} = \varepsilon.$$

Hence, f is not continuous at $(0, 0)$. □