Solutions to Assignment #10

1. An experiment consists of tossing a balanced die until a 6 comes up. On average, how many tosses are required to get a 6? In other words, if X denotes the number of tosses it takes to get a 6, what is $E(X)$? Show your calculations and justify your reasoning.

***Solution**:* If X denotes the number of tosses of the die that it takes to get a 6, then X has the pmf:

$$p_X(k) = \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6} \quad \text{for } k = 1, 2, 3, \ldots$$

We would like to compute

$$E(X) = \sum_{k=1}^{\infty} k p_X(k),$$

provided that it is a finite number.

We shall first do this problem for the general situation in which the probability of a 6 is p, for $0 < p < 1$. We then want to compute

$$E(X) = \sum_{k=1}^{\infty} k(1-p)^{k-1} \cdot p.$$

Now, for $0 < p < 1$, the geometric series

$$\sum_{k=0}^{\infty} (1-p)^k$$

converges to the function

$$f(p) = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

We can then write that

$$\sum_{k=0}^{\infty} (1-p)^k = f(p) \quad \text{for} \quad 0 < p < 1.$$

Differentiating with respect to p, we obtain that

$$\sum_{k=1}^{\infty} k(1-p)^{k-1} \cdot (-1) = f'(p) = -\frac{1}{p^2},$$

where we have used the chain rule when differentiating the summands. Thus, multiplying by $-p$ on both sides of the previous equation, we obtain that

$$\sum_{k=1}^{\infty} k(1-p)^{k-1} \cdot p = \frac{1}{p}.$$

Consequently,

$$E(X) = \frac{1}{p}.$$

In the special case in which $p = 1/6$ we obtain that $E(X) = 6$. Thus, on average, it takes 6 tosses of a balanced die to get a 6. □

2. Two discrete random variable, X and Y, are said to be **independent** if

$$\Pr(X = x,\ Y = y) = \Pr(X = x) \cdot \Pr(X = y)$$

for all possible values of x and y or X and Y, respectively.

Prove that if X and Y are discrete and independent, then

$$E(X + Y) = E(X) + E(Y).$$

Solution: Let $Z = X + Y$ and compute

$$\begin{aligned} E(X+Y) &= E(Z) \\ &= \sum_z z \cdot \Pr(X + Y = z) \\ &= \sum_z z \cdot \Pr(X = x, Y = z - x) \\ &= \sum_z z \cdot \Pr(X = x) \cdot \Pr(Y = z - x), \end{aligned}$$

by the independence of X and Y.

Thus, writing $x + y$ for z, where x denotes a value of X and y a value of Y,

$$\begin{aligned}
E(X+Y) &= \sum_x \sum_y (x+y)\cdot \Pr(X=x)\cdot \Pr(Y=y) \\
&= \sum_x \sum_y x\cdot \Pr(X=x)\cdot \Pr(Y=y) \\
&\qquad + \sum_x \sum_y y\cdot \Pr(X=x)\cdot \Pr(Y=y) \\
&= \sum_x x\cdot \Pr(X=x) \sum_y \Pr(Y=y) \\
&\qquad + \sum_x \Pr(X=x) \sum_y y\cdot \Pr(Y=y) \\
&= \sum_x x\cdot \Pr(X=x)\cdot 1 + \sum_x \Pr(X=x)\cdot E(Y) \\
&= E(X) + 1\cdot E(Y) \\
&= E(X) + E(Y).
\end{aligned}$$

□

3. Let X be a discrete random variable with pmf $p_X(x)$, and assume that $p_X(x)$ is positive at $x = -1, 0, 1$ and zero elsewhere.

   (a) If $p_X(0) = \dfrac{1}{4}$, find $E(X^2)$.

   Solution: First we find the pmf for X^2. The random variable X^2 takes on the values 0 and 1.
   Now,

$$\Pr(X^2 = 0) = \Pr(X = 0) = p_X(0) = \frac{1}{4},$$

   and

$$\begin{aligned}
\Pr(X^2 = 1) &= \Pr[(X=-1)\cup(X=1)] \\
&= p_X(-1) + p_X(1) \\
&= 1 - p_X(0) \\
&= \frac{3}{4}.
\end{aligned}$$

It then follows that

$$E(X^2) = 0 \cdot \Pr(X^2 = 0) + 1 \cdot \Pr(X^2 = 1) = \frac{3}{4}.$$

□

(b) If $p_X(0) = \dfrac{1}{4}$ and if $E(X) = \dfrac{1}{4}$, determine $p_X(-1)$ and $p_X(1)$.

Solution: From

$$E(X) = (-1) \cdot p_X(-1) + 0 \cdot p_X(0) + 1 \cdot p_X(1) = \frac{1}{4},$$

and

$$p_X(-1) + p_X(0) + \cdot p_X(1) = 1$$

we obtain the equations

$$\begin{cases} p_X(-1) - p_X(1) = -1/4, \\ p_X(-1) + p_X(1) = 3/4. \end{cases}$$

Solving these equations simultaneously yields

$$p_X(-1) = \frac{1}{4},$$

and

$$p_X(1) = \frac{1}{2}.$$

□

4. A bowl contains 10 chips, of which eight are marked \$2 and two are marked \$5 each. Let a person choose, at random and without replacement, three chips from the bowl. If the person is to receive the sum of the resulting amounts, find this expectation.

Solution: Let A denote the total amount of dollars marked in the there drawn chips. If X denotes the number of \$2 chips and Y the number of \$5 chips, then

$$A = 2X + 5Y.$$

Observe that $Y = 3 - X$; so that

$$A = 15 - 3X.$$

Then,

$$E(A) = 15 - 3E(X).$$

Thus, to find the expected value of A, we need to compute the expected value of X, where X takes on the values 1, 2 and 3, and its pmf is computed as follows

$$\Pr(X = 1) = \frac{\binom{8}{1} \cdot \binom{2}{2}}{\binom{10}{3}} = \frac{8}{120} = \frac{1}{15},$$

$$\Pr(X = 2) = \frac{\binom{8}{2} \cdot \binom{2}{1}}{\binom{10}{3}} = \frac{56}{120} = \frac{7}{15},$$

and

$$\Pr(X = 3) = \frac{\binom{8}{3} \cdot \binom{2}{0}}{\binom{10}{3}} = \frac{56}{120} = \frac{7}{15}.$$

It then follows that

$$\begin{aligned} E(X) &= 1 \cdot \Pr(X = 1) + 2 \cdot \Pr(X = 2) + 3 \cdot \Pr(X = 3) \\ &= \frac{1}{15} + \frac{14}{15} + \frac{21}{15} \\ &= \frac{36}{15} \\ &= \frac{12}{5}. \end{aligned}$$

We then have that

$$E(A) = 15 - 3 \cdot \frac{12}{5} = \frac{39}{5},$$

or \$7.80. □

5. Let $p_X(k) = \left(\dfrac{1}{2}\right)^k$, for $k = 1, 2, 3, \ldots$, zero elsewhere, be the pmf of a discrete random variable X. Find the mean value of X.

   Hint: For $|t| < 1$, define the function $f(t) = \displaystyle\sum_{k=0}^{\infty} t^k$. This is a geometric series which adds up to $\dfrac{1}{1-t}$. Compute $f'(t)$.

Solution: Differentiate the equation

$$\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}, \quad \text{for } |t| < 1,$$

with respect to t to obtain

$$\sum_{k=1}^{\infty} kt^{k-1} = \frac{1}{(1-t)^2}, \quad \text{for } |t| < 1.$$

Thus, multiplying by $t \neq 0$ on both sides we obtain

$$\sum_{k=1}^{\infty} kt^{k} = \frac{t}{(1-t)^2}, \quad \text{for } 0 < |t| < 1.$$

Applying this formula to the case $t = 1/2$ we obtain that

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} k p_X(k) \\ &= \sum_{k=1}^{\infty} k \left(\frac{1}{2}\right)^k \\ &= \frac{1/2}{(1-1/2)^2} \\ &= \frac{1/2}{1/4} \\ &= 2. \end{aligned}$$

That is, $E(X) = 2$. □