

Solutions to Assignment #11

1. An experiment consists of tossing a fair coin twice in a row. Let E_1 denote the event that a head comes up on the first toss, E_2 denote the event of heads on the second toss, and E_3 denote the event that exactly one head is thrown. Verify that E_1 , E_2 and E_3 are pairwise independent but that

$$\Pr(E_3 | E_1 \cap E_2) \neq \Pr(E_3).$$

Conclude therefore that the events E_1 , E_2 and E_3 are not mutually independent.

Solution: In this case the sample space consists of

$$\mathcal{C} = \{HH, HT, TH, TT\}.$$

Since the coins are fair, the events in \mathcal{C} are equally likely, so that $\Pr(c) = 1/4$ for all $c \in \mathcal{C}$.

We then have that $E_1 = \{HH, HT\}$, $E_2 = \{HH, TH\}$ and $E_3 = \{HT, TH\}$. Therefore,

$$\Pr(E_1) = \Pr(E_2) = \Pr(E_3) = \frac{1}{2}.$$

We also have that $E_1 \cap E_2 = \{HH\}$ so that

$$\Pr(E_1 \cap E_2) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \Pr(E_1) \cdot \Pr(E_2),$$

and so E_1 and E_2 are independent.

Similarly, since $E_1 \cap E_3 = \{HT\}$,

$$\Pr(E_1 \cap E_3) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \Pr(E_1) \cdot \Pr(E_3),$$

and so E_1 and E_3 are independent; and $E_2 \cap E_3 = \{TH\}$, so that

$$\Pr(E_2 \cap E_3) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \Pr(E_2) \cdot \Pr(E_3),$$

and so E_2 and E_3 are also independent.

On the other hand,

$$\Pr(E_3 | E_1 \cap E_2) = \frac{\Pr(E_3 \cap (E_1 \cap E_2))}{\Pr(E_1 \cap E_2)} = 0 \neq \Pr(E_3).$$

Consequently, E_1 , E_2 and E_3 are not mutually independent. \square

2. Let X have the pdf $f_X(x) = 3x^2$ for $0 < x < 1$, zero elsewhere. Consider a random rectangle whose sides are X and $1 - X$. Determine the expected value of the area of the rectangle.

Solution: The area of the rectangle is a random variable given by

$$A = X(1 - X) = X - X^2.$$

Then,

$$E(A) = E(X) - E(X^2),$$

where

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x) \, dx \\ &= \int_0^1 x \cdot 3x^2 \, dx \\ &= \frac{3}{4}, \end{aligned}$$

and

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f_X(x) \, dx \\ &= \int_0^1 x^2 \cdot 3x^2 \, dx \\ &= \frac{3}{5}. \end{aligned}$$

It then follows that

$$E(A) = \frac{3}{4} - \frac{3}{5} = \frac{3}{20}.$$

□

3. [Exercise 3 on page 188 in the text]

In a class of 50 students, the number of students n_i in each age class i is shown in the following Table 1.

If a student is to be selected at random from the class, what is the expected value of the student's age?

Age i	n_i
18	20
19	22
20	4
21	3
25	1

Table 1: Number of students in each age class

Solution: Define a random variable X that gives the age class a given student in the class is in; thus, $X = i$, where $i = 18, 19, 20, 21, 25$ years of age. Then, X has the distribution function:

$$p_X(i) = \begin{cases} 2/5 & \text{if } i = 18, \\ 11/25 & \text{if } i = 19, \\ 2/25 & \text{if } i = 20, \\ 3/50 & \text{if } i = 21, \\ 1/50 & \text{if } i = 25. \end{cases}$$

Then,

$$\begin{aligned} E(X) &= \sum_i i \cdot p_X(i) \\ &= 18 \frac{2}{5} + 19 \frac{11}{25} + 20 \frac{2}{25} + 21 \frac{3}{50} + 25 \frac{1}{50} \\ &= 18.92. \end{aligned}$$

□

4. [Exercise 4 on page 188 in the text]

Suppose that one word is to be selected at random from the sentence

THE GIRL PUT ON HER BEAUTIFUL RED HAT.

If X denotes the number of letters in the word that is selected, what is the value of $E(X)$?

Solution: X takes on the values 2, 3, 4 and 9. The distribution for X is then

$$p_X(k) = \begin{cases} 1/8 & \text{if } k = 2, \\ 5/8 & \text{if } k = 3, \\ 1/8 & \text{if } k = 4, \\ 1/8 & \text{if } k = 9. \end{cases}$$

The expected value of X is then

$$\begin{aligned} E(X) &= \sum_k k \cdot p_X(k) \\ &= 2\frac{1}{8} + 3\frac{5}{8} + 4\frac{1}{8} + 9\frac{1}{8} \\ &= 3.75. \end{aligned}$$

□

5. [Exercise 5 on page 188 in the text]

Suppose that one letter is to be selected at random from the 30 letters in the sentence given in the previous exercise. If Y denotes the number of letters in the word in which the selected letter appears, what is the value of $E(Y)$?

Solution: Y takes on the values 2, 3, 4 and 9. The distribution for Y is computed as follows:

$$\Pr(Y = 2) = \frac{2}{30} = \frac{1}{15},$$

since only 2 of the 30 letters are in a 2-letter word. Similarly,

$$\Pr(Y = 3) = \frac{15}{30} = \frac{1}{2},$$

since 15 of the 30 letters are in 3-letter words. Continuing in this fashion we get the pmf

$$p_Y(k) = \begin{cases} 1/15 & \text{if } k = 2, \\ 1/2 & \text{if } k = 3, \\ 2/15 & \text{if } k = 4, \\ 3/10 & \text{if } k = 9. \end{cases}$$

The expected value of Y is then

$$\begin{aligned} E(Y) &= \sum_k k \cdot p_Y(k) \\ &= 2\frac{1}{15} + 3\frac{1}{2} + 4\frac{2}{15} + 9\frac{3}{10} \\ &= 4.867. \end{aligned}$$

□