Solutions to Assignment #12

1. A balanced die is tossed n times. Let X denote the number of 1's that come up. Give the pmf for X and compute its expectation.

   ***Solution**:* The probability of a success in each independent trial in this case is $p = 1/6$. Thus,

$$X \sim \text{Binomial}(n, 1/6),$$

   so that

$$p_X(k) = \binom{n}{k}\left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n,$$

   and

$$E(X) = \frac{n}{6}.$$

   □

2. Let X and Y denote independent Binomial(n, p) random variables and put $Z = X + Y$. Determine the pmf of Z and compute its expectation.

   Hint: Suppose there are n red balls and n blue balls in a box. Compute the number of ways of picking k balls out of the box, l of which are red and $k - l$ of which are blue.

   ***Solution**:* The pmf for X and Y are, respectively,

$$p_X(\ell) = \binom{n}{\ell} p^\ell (1-p)^{n-\ell} \quad \text{for } \ell = 0, 1, 2, \ldots, n,$$

   and

$$p_Y(m) = \binom{n}{m} p^m (1-p)^{n-m} \quad \text{for } m = 0, 1, 2, \ldots, n.$$

   Using the independence of X and Y, we compute

$$\begin{aligned} p_Z(k) &= \Pr(Z = k) \\ &= \Pr(X + Y = k) \\ &= \sum_{\ell=0}^{k} \Pr(X = \ell, Y = k - \ell), \end{aligned}$$

since the event $(X + Y = k)$ is the disjoint union of the events

$$(X = \ell, Y = k - \ell)$$

for $\ell = 0, 1, 2, \ldots, k$. Thus,

$$\begin{aligned} p_{_Z}(k) &= \sum_{\ell=0}^{k} \Pr(X = \ell) \cdot \Pr(Y = k - \ell) \\ &= \sum_{\ell=0}^{k} \binom{n}{\ell} p^{\ell}(1-p)^{n-\ell} \cdot \binom{n}{k-\ell} p^{k-\ell}(1-p)^{n-k+\ell} \\ &= \sum_{\ell=0}^{k} \binom{n}{\ell}\binom{n}{k-\ell} p^{k}(1-p)^{2n-k}. \end{aligned}$$

Now, $\displaystyle\sum_{\ell=0}^{k} \binom{n}{\ell}\binom{n}{k-\ell}$ counts the number of ways of picking k balls out of $2n$ where n are red and n are blue; that is,

$$\sum_{\ell=0}^{k} \binom{n}{\ell}\binom{n}{k-\ell} = \binom{2n}{k}.$$

It then follows that

$$p_{_Z}(k) = \binom{2n}{k} p^{k}(1-p)^{2n-k}, \quad \text{for } k = 0, 1, 2, \ldots, n.$$

Thus, $Z = X + Y$ as a binomial distribution with parameters $2n$ and p. We therefore get that

$$E(Z) = 2np.$$

□

3. *(Random Walk on the Integers).* A particle starts at $x = 0$ and, after one unit of time, it moves one unit to the right with probability p, for $0 < p < 1$, or to the left with probability $1 - p$. Let X_1 denote the position of the particle after one unit of time and X_2 denote that after 2 units of time. Give the probability mass functions for X_1 and X_2 and compute their expectations. Assume that at each time step, whether a particle will move to the right or to the left is independent of where it has been.

***Solution**:* X_1 can take on the values 1 or -1 depending on whether the particle moves to the right or to the left, respectively. We then have that

$$p_{X_1}(x) = \begin{cases} 1-p & \text{if } x = -1, \\ p & \text{if } x = 1. \end{cases}$$

Consequently,

$$E(X_1) = (-1)(1-p) + (1)p = 2p - 1.$$

Similarly, X_2 can take on the values -2, 0 or 2, and its pmf is

$$p_{X_2}(x) = \begin{cases} (1-p)^2 & \text{if } x = -2, \\ 2p(1-p) & \text{if } x = 0, \\ p^2 & \text{if } x = 2. \end{cases}$$

In the first case the particle moves to the left twice, each time with probability $1-p$. In the second case, the particle could get back to 0 by first moving to the right and the to the left (with probability $p(1-p)$) or first to the left and then to the right (with probability $(1-p)p$. Finally in the third case the particle moves to the right twice in a row with probability p^2.
The expected value of X_2 is then

$$E(X_2) = (-2)(1-p)^2 + (0)2p(1-p) + 2p^2 = 2[p^2 - (1-p)^2] = 2(2p-1).$$

□

4. *(Random Walk on the Integers, Continued).* Let X_3 denote the position of the particle in the previous problem after 3 units of time. Give its pmf and expectation. Generalize this result to X_n, the position of the particle after n units of time.

***Solution**:* Observe that in the previous problem $X_2 = X_1 + X_1$. We can thus obtain X_3 by adding X_1 to X_2; that is,

$$X_3 = X_2 + X_1,$$

where X_2 and X_1 are independent. We then have that X_3 can take on the values -3, -1, 1 or 3.

We compute

$$\begin{aligned}
\Pr(X_3 = -3) &= \Pr(X_2 = -2, X_1 = -1) \\
&= \Pr(X_2 = -2) \cdot \Pr(X_1 = -1) \\
&= (1-p)^2(1-p) \\
&= (1-p)^3
\end{aligned}$$

and

$$\begin{aligned}
\Pr(X_3 = -1) &= \Pr(X_2 = -2, X_1 = 1) + \Pr(X_2 = 0, X_1 = -1) \\
&= \Pr(X_2 = -2) \cdot \Pr(X_1 = 1) + \Pr(X_2 = 0) \cdot \Pr(X_1 = -1) \\
&= (1-p)^2 p + 2p(1-p)(1-p) \\
&= 3p(1-p)^2.
\end{aligned}$$

Similarly, we can compute

$$\begin{aligned}
\Pr(X_3 = 1) &= \Pr(X_2 = 2, X_1 = -1) + \Pr(X_2 = 0, X_1 = 1) \\
&= \Pr(X_2 = 2) \cdot \Pr(X_1 = -1) + \Pr(X_2 = 0) \cdot \Pr(X_1 = 1) \\
&= p^2(1-p) + 2p(1-p)p \\
&= 3p^2(1-p),
\end{aligned}$$

and

$$\begin{aligned}
\Pr(X_3 = 3) &= \Pr(X_2 = 2, X_1 = 1) \\
&= \Pr(X_2 = 2) \cdot \Pr(X_1 = 1) \\
&= p^2 p \\
&= p^3.
\end{aligned}$$

We then have that the pmf for X_3 is

$$p_{X_3}(x) = \begin{cases} (1-p)^3 & \text{if } x = -3, \\ 3p(1-p)^2 & \text{if } x = -1, \\ 3p^2(1-p) & \text{if } x = 1, \\ p^3 & \text{if } x = 3. \end{cases}$$

To compute the expected value of X_3 we may use the fact the $X_3 = X_2 + X_1$ so that

$$E(X_3) = E(X_2) + E(X_1) = 2(2p-1) + 2p - 1 = 3(2p-1).$$

The previous calculations suggest that the pmf for X_4 must be

$$p_{X_4}(x) = \begin{cases} (1-p)^4 & \text{if } x = -4, \\ 4p(1-p)^3 & \text{if } x = -2, \\ 6p^2(1-p)^2 & \text{if } x = 0, \\ 4p^3(1-p) & \text{if } x = 2, \\ p^4 & \text{if } x = 4, \end{cases}$$

and its expected value is

$$E(X_4) = 4(2p - 1).$$

This suggests that the pmf of X_n is

$$p_{X_n}(2k - n) = \binom{n}{k} p^k (1 - p)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n,$$

and its expectation is

$$E(X_n) = n(2p - 1).$$

This result can be established by induction on n. □

5. Toss a coin 100 times, and let X denote the number of heads that come up. Given that the probability of a head is p, where $0 < p < 1$, give the distribution function of X and compute $\Pr(35 \leqslant X \leqslant 45)$ for the cases $p = 0.5$ and $p = 0.4$.

***Solution**:* In this case, $X \sim \text{Binomial}(100, p)$ so that

$$p_X(k) = \binom{100}{k} p^k (1 - p)^{100-k} \quad \text{for } k = 0, 1, 2, \ldots, 100.$$

Therefore

$$\Pr(35 \leqslant X \leqslant 45) = \sum_{k=35}^{4} 5p_X(k),$$

which can also be computed using the cumulative distribution function as follows

$$\Pr(35 \leqslant X \leqslant 45) = F_X(45) - F_X(34).$$

Using the `binomdist` function in MS Excel we obtain

$$\Pr(35 \leqslant X \leqslant 45) \approx 0.1832 \text{ or } 18.32\% \text{ for } p = 0.5,$$

and

$$\Pr(35 \leqslant X \leqslant 45) \approx 0.7386 \text{ or } 73.86\% \text{ for } p = 0.4.$$

□