Solutions to Assignment #13

1. Compute the moment generating function, $\psi(t)$, of a continuous random variable X with Uniform$(-1, 2)$ distribution. What should $\psi(0)$ be? Give also the second moment and variance of X.

***Solution**:* The pdf of X is given by

$$f_X(x) = \begin{cases} \dfrac{1}{3} & \text{if } \quad -1 < x < 2, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Then, the mgf of

$$\begin{aligned} \psi_X(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\, \mathrm{d}x \\ &= \int_{-1}^{2} e^{tx} \frac{1}{3}\, \mathrm{d}x \\ &= \frac{1}{3} \cdot \frac{e^{2t} - e^{-t}}{t} \end{aligned}$$

provided that $t \neq 0$. It $t = 0$, then $\psi_X(t) = \psi_X(0) = E(1) = 1$. We then have that

$$\psi_X(t) = \begin{cases} \dfrac{e^{2t} - e^{-t}}{3t} & \text{if } t \neq 0, \\ \\ 1 & \text{if } t = 0. \end{cases}$$

Next, compute the expectation and second moment of X:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, \mathrm{d}x = \frac{1}{3} \int_{-1}^{2} x\, \mathrm{d}x = \frac{1}{2},$$

and

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, \mathrm{d}x = \frac{1}{3} \int_{-1}^{2} x^2\, \mathrm{d}x = 1.$$

Then, the variance of X is

$$\text{var}(X) = E(X^2) - (E(X))^2 = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}.$$

□

2. *[Exercise 2 on page 203 in the text]*

   Suppose that one word is selected at random from the sentence

   THE GIRL PUT ON HER BEAUTIFUL HAT.

   If X denotes the number of letters in the word that is selected, what is the value of $\text{var}(X)$?

   Solution: X takes on the values 2, 3, 4 and 9. The distribution for X is then

$$p_X(k) = \begin{cases} 1/8 & \text{if } k = 2, \\ 5/8 & \text{if } k = 3, \\ 1/8 & \text{if } k = 4, \\ 1/8 & \text{if } k = 9. \end{cases}$$

   The expected value of X is then

$$\begin{aligned} E(X) &= \sum_k k \cdot p_X(k) \\ &= 2\frac{1}{8} + 3\frac{5}{8} + 4\frac{1}{8} + 9\frac{1}{8} \\ &= 3.75. \end{aligned}$$

   The second moment of X is

$$\begin{aligned} E(X^2) &= \sum_k k^2 \cdot p_X(k) \\ &= 4\frac{1}{8} + 9\frac{5}{8} + 16\frac{1}{8} + 81\frac{1}{8} \\ &= 18.25. \end{aligned}$$

   Thus, the variance of X is

$$\text{var}(X) = E(X^2) - (E(X))^2 = 18.25 - (3.75)^2 = 4.1875.$$

   □

3. *[Exercise 4 on page 203 in the text]*

   Suppose that X is a random variable for which $E(X) = \mu$ and $\text{var}(X) = \sigma^2$. Show that

   $$E[X(X-1)] = \mu(\mu-1) + \sigma^2.$$

   ***Solution**:* Compute

   $$\begin{aligned} E[X(X-1)] &= E(X^2 - X) \\ &= E(X^2) - E(X) \\ &= E(X^2) - \mu^2 + \mu^2 - E(X) \\ &= \text{var}(X) + \mu^2 - \mu \\ &= \sigma^2 + \mu(\mu-1), \end{aligned}$$

   which is what we were asked to show. □

4. *[Exercise 8 on page 209 in the text]*

   Suppose that X is a random variable for which the mgf is as follows:

   $$\psi_X(t) = e^{t^2+3t} \quad \text{for } -\infty < t < \infty.$$

   Find the mean and variance of X.

   ***Solution**:* Taking derivatives we get

   $$\psi_X'(t) = (2t+3)e^{t^2+3t} \quad \text{for } -\infty < t < \infty,$$

   and

   $$\psi_X''(t) = (2t+3)^2 e^{t^2+3t} + 2e^{t^2+3t} \quad \text{for } -\infty < t < \infty.$$

   Then

   $$\mu = E(X) = \psi_X'(0) = 3,$$

   and

   $$E(X^2) = \psi_X''(0) = 11.$$

   Consequently, the variance of X is

   $$\text{var}(X) = E(X^2) - \mu^2 = 11 - 9 = 2.$$

   □

5. *[Exercise 12 on page 209 in the text]*

   Suppose that X is a random variable for which the mgf is as follows:

$$\psi_X(t) = \frac{1}{6}(4 + e^t + e^{-t}) \quad \text{for } -\infty < t < \infty.$$

   Find the probability distribution of X.

   ***Solution**:* Consider a discrete random variable, X, with three possible values x_1, x_2 and x_3, and corresponding probabilities p_1, p_2 and p_3 (so that $p_1 + p_2 + p_3 = 1$). Then its mgf is given by

$$\psi_X(t) = E(e^{tX}) = e^{x_1 t}p_1 + e^{x_2 t}p_2 + e^{x_3 t}p_3.$$

   Comparing this to the given mgf,

$$\psi_X(t) = \frac{1}{6}e^{-t} + \frac{2}{3} + \frac{1}{6}e^{t} \quad \text{for } -\infty < t < \infty,$$

   we see that $x_1 = -1$, $x_2 = 0$ and $x_3 = 1$, and the pmf of X is then

$$p_X(x) = \begin{cases} 1/6 & \text{if } x = -1, \\ 2/3 & \text{if } x = 0, \\ 1/6 & \text{if } x = 1. \end{cases}$$

$\square$