Solutions to Assignment #14

1. *[Exercise 2 on page 127 in the text]*

   Suppose that in an electric display sign there are three light bulbs in the first row and four light bulbs in the second row. Let X denote the number of bulbs in the first row that will be burned out at a specified time t, and let Y denote the number of bulbs in the second row that will be burned out at the same time t. Suppose that the joint pmf of X and Y is as specified in Table 1:

$X\backslash Y$	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

Table 1: Joint Probability Distribution for X and Y, $p_{(X,Y)}$

Determine each of the following probabilities:

(a) $\Pr(X = 2)$

(b) $\Pr(Y \geqslant 2)$

(c) $\Pr(X \leqslant 2 \text{ and } Y \leqslant 2)$

(d) $\Pr(X = Y)$

(e) $\Pr(X > Y)$

Solution:

(a) Add probabilities along the third row:

$$\Pr(X = 2) = 0.05 + 0.06 + 0.09 + 0.04 + 0.03 = 0.27.$$

(b) Compute

$$\Pr(Y \geqslant 2) = \sum_{j=2}^{4} \Pr(Y = j).$$

This is the sum of the probabilities on the last three columns; thus,

$$\Pr(Y \geqslant 2) = 0.53.$$

(c) Add entries on the first three rows and columns:

$$\Pr(X \leqslant 2 \text{ and } Y \leqslant 2) = 0.69.$$

(d) Add entries along the "main diagonal:"

$$\Pr(X = Y) = 0.08 + 0.10 + 0.09 + 0.03 = 0.30.$$

(e) Add entries below the "main diagonal:"

$$\Pr(X > Y) = 0.06 + 0.05 + 0.02 + 0.06 + 0.03 + 0.03 = 0.23.$$

□

2. *[Exercise 4 on page 127 in the text]*

Suppose that X and Y have a continuous joint distribution for which the pdf is defined as follows:

$$f(x,y) = \begin{cases} cy^2 & \text{for } 0 \leqslant x \leqslant 2 \text{ and } 0 \leqslant y \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine

(a) the value of the constant c;

(b) $\Pr(X + Y > 2)$;

(c) $\Pr(Y < 1/2)$;

(d) $\Pr(X \leqslant 1)$;

(e) $\Pr(X = 3Y)$.

***Solution**:*

(a) We find c so that $\iint_{\mathbb{R}^2} f(x,y)\, \mathrm{d}x\mathrm{d}y = 1$; that is, so that

$$c \int_0^1 \int_0^2 y^2 \, \mathrm{d}x\mathrm{d}y = 1,$$

or

$$c\frac{2}{3} = 1,$$

from which we get that $c = 3/2$.

(b) Compute

$$\Pr(X + Y > 2) = \iint_A f(x, y)\, \mathrm{d}x\mathrm{d}y,$$

where

$$A = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1,\ x + y > 2\}.$$

The region A is sketched in Figure 1:

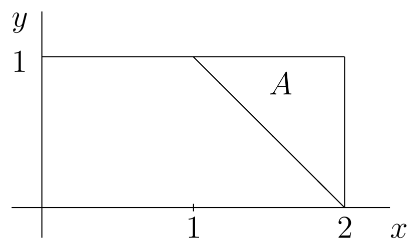

Figure 1: Sketch of region A

We then have that

$$\begin{aligned}\Pr(X + Y > 2) &= \int_0^1 \int_{2-y}^2 \frac{3}{2} y^2\, \mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \frac{3}{2} y^2\, x\Big|_{2-y}^{2}\, \mathrm{d}y \\ &= \int_0^1 \frac{3}{2} y^3 \mathrm{d}y \\ &= \frac{3}{8}.\end{aligned}$$

(c) Compute

$$\Pr(Y < 1/2) = \iint_A f(x, y)\, \mathrm{d}x\mathrm{d}y,$$

where

$$A = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1/2\}.$$

The region A is sketched in Figure 2:

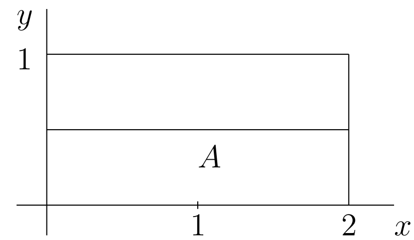

Figure 2: Sketch of region A

We then have that

$$\begin{aligned}\Pr(Y > 1/2) &= \int_0^{1/2}\int_0^2 \frac{3}{2}y^2 \,\mathrm{d}x\mathrm{d}y \\ &= \int_0^{1/2} \frac{3}{2}y^2 \, x\Big|_0^2 \,\mathrm{d}y \\ &= \int_0^{1/2} 3y^2\mathrm{d}y \\ &= y^3\Big|_0^{1/2} \\ &= \frac{1}{8}.\end{aligned}$$

(d) Compute

$$\Pr(X \leqslant 1) = \iint_A f(x,y)\,\mathrm{d}x\mathrm{d}y,$$

where

$$A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}.$$

The region A is sketched in Figure 3:

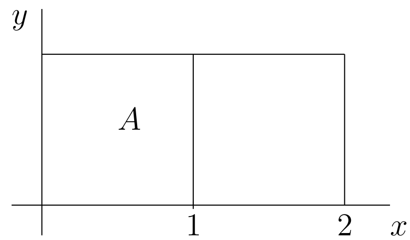

Figure 3: Sketch of region A

We then have that

$$
\begin{aligned}
\Pr(Y > 1/2) &= \int_0^1 \int_0^1 \frac{3}{2} y^2 \, \mathrm{d}x\mathrm{d}y \\
&= \int_0^1 \frac{3}{2} y^2 \, x \Big|_0^1 \, \mathrm{d}y \\
&= \int_0^1 \frac{3}{2} y^2 \mathrm{d}y \\
&= \frac{1}{2} y^3 \Big|_0^1 \\
&= \frac{1}{2}.
\end{aligned}
$$

(e) $\Pr(X = 3Y) = 0$.

□

3. *[Exercise 6 on page 127 in the text]*

   Suppose a point X is chosen at random from a region S in the xy–plane containing all points (x, y) such that $x \geqslant 0$, $y \geqslant 0$, and $4y + x \leqslant 4$.

   (a) Determine the joint pdf of X and Y.

   (b) Suppose that S_o is a subset of the region S having area α, and determine $\Pr[(X, Y) \in S_o]$.

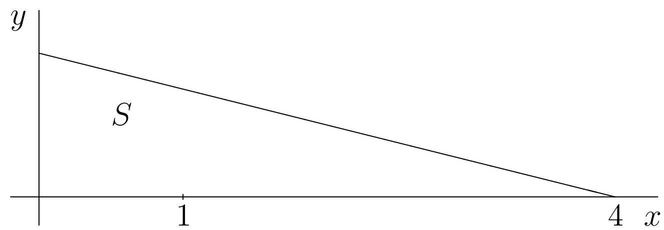

Figure 4: Sketch of region A

Solution:

(a) The region S is sketched in Figure 4:
We want (X,Y) to have uniform distribution on S. We therefore define the joint distribution pdf of X and Y to be:

$$f_{(X,Y)}(x,y) = \begin{cases} \dfrac{1}{2} & \text{if } (x,y) \in S, \\ \\ 0 & \text{otherwise.} \end{cases}$$

(b) Compute

$$\begin{aligned} \Pr[(X,Y) \in S_o] &= \iint_{S_o} f_{(X,Y)}(x,y)\, \mathrm{d}x\mathrm{d}y \\ &= \iint_{S_o} \frac{1}{2}\, \mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2} \iint_{S_o} \mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2} \cdot \text{area}(S_o) \\ &= \frac{\alpha}{2}. \end{aligned}$$

□

4. *[Exercise 2 on page 135 in the text]*

   Suppose that X and Y have a discrete distribution for which the joint pmf is defined as follows:

$$p_{(X,Y)}(x,y) = \begin{cases} \frac{1}{30}(x+y) & \text{for } x = 0,1,2 \text{ and } y = 0,1,2,3, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Determine the marginal pmfs of X and Y.

   (b) Are X and Y independent?

   ***Solution**:*

   (a) The marginal pmf of X is

$$\begin{aligned} p_X(x) &= \sum_{y=0}^{3} p_{(X,Y)}(x,y) \\ &= \frac{1}{30}\sum_{y=0}^{3}(x+y) \\ &= \frac{1}{30}\sum_{y=0}^{3} x + \frac{1}{30}\sum_{y=0}^{3} y \\ &= \frac{4}{30}x + \frac{6}{30} \\ &= \frac{2}{15}x + \frac{1}{5}, \end{aligned}$$

   for $x = 0, 1, 2$.

Similarly, the marginal pmf of Y is

$$\begin{aligned} p_Y(y) &= \sum_{x=0}^{2} p_{(X,Y)}(x,y) \\ &= \frac{1}{30}\sum_{x=0}^{2}(x+y) \\ &= \frac{1}{30}\sum_{x=0}^{2} x + \frac{1}{30}\sum_{x=0}^{2} y \\ &= \frac{3}{30} + \frac{3}{30}y \\ &= \frac{1}{10} + \frac{1}{10}y, \end{aligned}$$

for $y = 0, 1, 2, 3$.

(b) Observe that

$$p_X(x)\cdot p_Y(y) = \frac{1}{150}(2x+3)(y+1), \text{ for } x = 0, 1, 2 \text{ and } y = 0, 1, 2, 3,$$

which is not the same as $p_{(X,Y)}(x,y)$. It then follows that X and Y are not independent.

□

5. *[Exercise 4 on page 136 in the text]*

Suppose the joint pdf of X and Y is as follows:

$$f_{(X,Y)}(x,y) = \begin{cases} \frac{15}{4}x^2 & \text{for } 0 \leqslant y \leqslant 1 - x^2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the marginal pdfs of X and Y.

(b) Are X and Y independent?

***Solution**:*

(a) The marginal distribution of X is

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\,\mathrm{d}y \\ &= \int_0^{1-x^2} \frac{15}{4}x^2\,\mathrm{d}y, \end{aligned}$$

for $-1 < x < 1$. (see Figure 5).

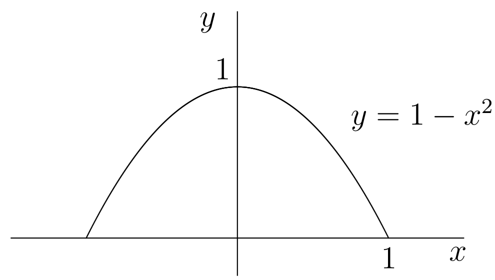

Figure 5: Sketch of region

Thus,

$$f_X(x) = \begin{cases} \dfrac{15}{4}x^2(1-x^2) & \text{if } -1 < x < 1, \\ \\ 0 & \text{otherwise.} \end{cases}$$

To find the marginal distribution of Y consider Figure 6:

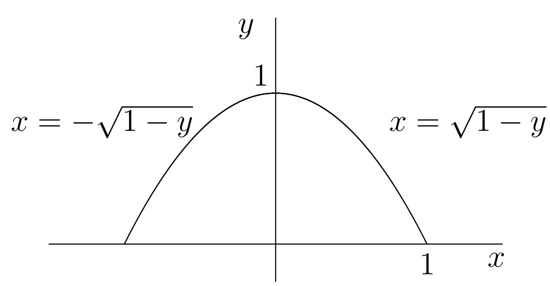

Figure 6: Sketch of region

$$
\begin{aligned}
f_{Y}(y) &= \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\ \mathrm{d}x \\
&= \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{15}{4}x^2\ \mathrm{d}x \\
&= 2\int_{0}^{\sqrt{1-y}} \frac{15}{4}x^2\ \mathrm{d}x \\
&= \frac{5}{2}(1-y)^{3/2},
\end{aligned}
$$

for $0 < y < 1$.
It then follows that

$$
f_{Y}(y) = \begin{cases} \dfrac{5}{2}(1-y)^{3/2} & \text{if } 0 < y < 1, \\ \\ 0 & \text{otherwise.} \end{cases}
$$

(b) Observe that

$$
f_{X}(x)\cdot f_{Y}(y) = \frac{75}{8}x^2(1-x^2)(1-y)^{3/2},
$$

which is not the same as the given joint pdf. Consequently, X and Y are not independent.

□