Solutions to Assignment #15

1. Let $F_{(X,Y)}$ be the joint cdf of two random variables X and Y. For real constants $a < b$, $c < d$, show that

$$\Pr(a < X \leqslant b, c < Y \leqslant d) = F_{(X,Y)}(b,d) - F_{(X,Y)}(b,c) - F_{(X,Y)}(a,d) + F_{(X,Y)}(a,c).$$

Use this result to show that $F(x,y) = \begin{cases} 1 & \text{if } x + 2y \geqslant 1, \\ 0 & \text{otherwise,} \end{cases}$ cannot be the joint cdf of two random variables.

***Solution**:* Let $A = \{(x,y) \in \mathbb{R}^2 \mid a < x \leqslant b,\ c < y \leqslant d\}$; we then want to compute $\Pr[(X,Y) \in A]$. In addition, define the events:

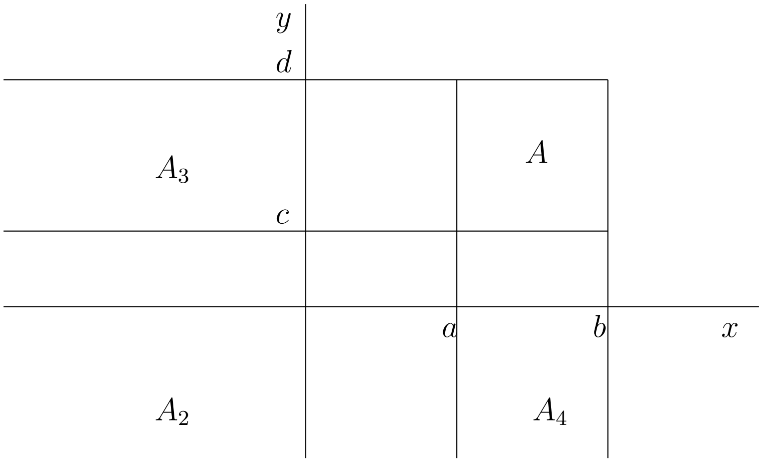

Figure 1: Events A, A_1, A_2, A_3 and A_4 in the xy–plane

$$A_1 = \{(x,y) \in \mathbb{R}^2 \mid x \leqslant b,\ y \leqslant d\},$$

$$A_2 = \{(x,y) \in \mathbb{R}^2 \mid x \leqslant a,\ y \leqslant c\},$$

$$A_3 = \{(x,y) \in \mathbb{R}^2 \mid x \leqslant a,\ c < y \leqslant d\},$$

and

$$A_4 = \{(x,y) \in \mathbb{R}^2 \mid a < x \leqslant b,\ y \leqslant c\}.$$

Then, A_1 is a disjoint union of the events A, A_2, A_3 and A_4 (see Figure 1). It then follows that

$$\Pr[(X,Y) \in A_1] = \Pr[(X,Y) \in A \cup A_2 \cup A_3 \cup A_4]$$

or

$$\begin{aligned} \Pr[(X,Y)\in A_1] &= \Pr[(X,Y)\in A]+\Pr[(X,Y)\in A_2]+ \\ &\quad +\Pr[(X,Y)\in A_3]+\Pr[(X,Y)\in A_4]. \end{aligned} \tag{1}$$

Observe that

$$\Pr[(X,Y)\in A_1]=\Pr(X\leqslant b, Y\leqslant d)=F_{(X,Y)}(b,d)$$

and

$$\Pr[(X,Y)\in A_2]=\Pr(X\leqslant a, Y\leqslant c)=F_{(X,Y)}(a,c).$$

It then follows from equation (1) that

$$\begin{aligned} \Pr[(X,Y)\in A] &= F_{(X,Y)}(b,d)-F_{(X,Y)}(a,c) \\ &\quad -\Pr[(X,Y)\in A_3]-\Pr[(X,Y)\in A_4]. \end{aligned} \tag{2}$$

On the other hand, observe that

$$\Pr[(X,Y)\in A_3\cup A_2]=\Pr(X\leqslant a, Y\leqslant d)=F_{(X,Y)}(a,d) \tag{3}$$

and

$$\Pr[(X,Y)\in A_4\cup A_2]=\Pr(X\leqslant b, Y\leqslant c)=F_{(X,Y)}(b,c). \tag{4}$$

Moreover,

$$\begin{aligned} \Pr[(X,Y)\in (A_3\cup A_2)\cup(A_4\cup A_2)] &= \Pr[(X,Y)\in A_3\cup A_2] \\ &\quad +\Pr[(X,Y)\in A_4\cup A_2] \\ &\quad -\Pr[(X,Y)\in A_2], \end{aligned}$$

since $(A_3\cup A_2)\cap(A_4\cup A_2)=A_2$. It then follows from equations (3) and (4) that

$$\begin{aligned} \Pr[(X,Y)\in (A_3\cup A_2)\cup(A_4\cup A_2)] &= F_{(X,Y)}(a,d) \\ &\quad +F_{(X,Y)}(b,c) \\ &\quad -F_{(X,Y)}(a,c). \end{aligned}$$

However, since $(A_3\cup A_2)\cup(A_4\cup A_2)=A_2\cup A_3\cup A_4$, we also get that

$$\begin{aligned} \Pr[(X,Y)\in (A_3\cup A_2)\cup(A_4\cup A_2)] &= \Pr[(X,Y)\in A_2] \\ &\quad +\Pr[(X,Y)\in A_3] \\ &\quad +\Pr[(X,y)\in A_4]. \end{aligned}$$

We therefore get, using $\Pr[(X,Y)\in A_2]=F_{(X,Y)}(a,c)$, that

$$\begin{aligned}\Pr[(X,Y)\in A_3]+\Pr[(X,Y)\in A_4] &= F_{(X,Y)}(a,d)+F_{(X,Y)}(b,c)\\ &\qquad -2F_{(X,Y)}(a,c).\end{aligned}$$

Substituting this into equation (2) yields

$$\begin{aligned}\Pr[(X,Y)\in A] &= F_{(X,Y)}(b,d)-F_{(X,Y)}(a,c)\\ &\quad -F_{(X,Y)}(a,d)-F_{(X,Y)}(b,c)+2F_{(X,Y)}(a,c),\end{aligned}$$

from which we get

$$\Pr[(X,Y)\in A]=F_{(X,Y)}(b,d)-F_{(X,Y)}(a,d)-F_{(X,Y)}(b,c)+F_{(X,Y)}(a,c).$$

Next, suppose that $F(x,y)=\begin{cases}1 & \text{if } x+2y\geqslant 1,\\ 0 & \text{otherwise,}\end{cases}$ is the joint cdf of two random variables X and Y. Consider the set

$$A=\{(x,y)\in\mathbb{R}^2\mid 0<x\leqslant 1, 0<y\leqslant 1/2\}.$$

By what we just proved,

$$\begin{aligned}\Pr[(X,Y)\in A] &= F(1,1/2)-F(0,1/2)-F(1,0)+F(0,0)\\ &= 1-1-1+0\\ &= -1<0,\end{aligned}$$

which is impossible since $\Pr[(X,Y)\in A]\geqslant 0$. Therefore, F cannot be a joint pdf.

□

2. Let $g(t)$ denote a non–negative, integrable function of a single variable with the property that

$$\int_0^\infty g(t)\ \mathrm{d}t=1.$$

Define

$$f(x,y)=\begin{cases}\dfrac{2g(\sqrt{x^2+y^2})}{\pi\sqrt{x^2+y^2}} & \text{for } 0<x<\infty,\ 0<y<\infty,\\ \\ 0\text{otherwise.}\end{cases}$$

Show that $f(x,y)$ is a joint pdf for two random variables X and Y.

***Solution**:* First observe that f is non–negative since g is non–negative. Next, compute

$$\iint_{\mathbb{R}^2} f(x,y)\ \mathrm{d}x\mathrm{d}y \quad = \quad \int_0^\infty \int_0^\infty \frac{2g(\sqrt{x^2+y^2})}{\pi\sqrt{x^2+y^2}}\ \mathrm{d}x\mathrm{d}y.$$

Switching to polar coordinates we then get that

$$\begin{aligned} \iint_{\mathbb{R}^2} f(x,y)\ \mathrm{d}x\mathrm{d}y &= \int_0^{\pi/2} \int_0^\infty \frac{2g(r)}{\pi r}\ r\mathrm{d}r\mathrm{d}\theta \\ &= \frac{\pi}{2} \int_0^\infty \frac{2}{\pi} g(r)\ \mathrm{d}r \\ &= \int_0^\infty g(r)\ \mathrm{d}r \\ &= 1, \end{aligned}$$

and therefore $f(x,y)$ is indeed a joint pdf for two random variables X and Y. □

3. Let X and Y have joint pdf

$$f_{(X,Y)}(x,y) = \begin{cases} e^{-x-y} & \text{for } 0 < x < \infty,\ 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Define $Z = X + Y$. Compute $\Pr(Z \leqslant z)$ for $0 < z < \infty$ and give the pdf of Z.

***Solution**:*

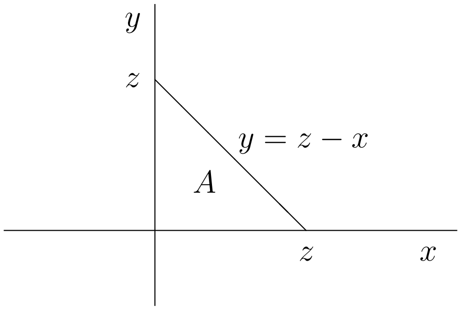

Figure 2: Event A in the xy–plane

Compute

$$\begin{aligned}\Pr(Z \leqslant z) &= \Pr(X+Y \leqslant z), \quad \text{for } z > 0\\ &= \iint_A f_{(X,Y)}(x,y)\ \mathrm{d}x\mathrm{d}y,\end{aligned}$$

where $A = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0, x + y \leqslant z\}$ (see Figure 2). We then have that

$$\begin{aligned}\Pr(Z \leqslant z) &= \int_0^z \int_0^{z-x} e^{-x-y}\ \mathrm{d}y\mathrm{d}x\\ &= \int_0^z e^{-x} \int_0^{z-x} e^{-y}\ \mathrm{d}y\mathrm{d}x\\ &= \int_0^z e^{-x}\left[-e^{-y}\right]_0^{z-x}\ \mathrm{d}x\\ &= \int_0^z e^{-x}(1-e^{-(z-x)})\ \mathrm{d}x\\ &= \int_0^z (e^{-x}-e^{-z})\ \mathrm{d}x\\ &= \left[-e^{-x}\right]_0^z - ze^{-z}\\ &= 1 - e^{-z} - ze^{-z}.\end{aligned}$$

Thus, $F_Z(z) = 1 - e^{-z} - ze^{-z}$ for all $z > 0$. It then follows that the pdf of Z is

$$f_Z(z) = \begin{cases} ze^{-z} & \text{if } z > 0,\\ 0 & \text{otherwise.}\end{cases}$$

$\square$

4. Let X and Y have joint pdf

$$f_{(X,Y)}(x,y) = \begin{cases} 1 & \text{for } 0 < x < 1,\ 0 < y < 1,\\ 0 & \text{otherwise.}\end{cases}$$

Find the cdf and pdf of the product $Z = XY$.

Solution: Compute

$$\begin{aligned}\Pr(Z \leqslant z) &= \Pr(XY \leqslant z), \quad \text{for } 0 < z < 1 \\ &= \iint_A f_{(X,Y)}(x, y)\, \mathrm{d}x\mathrm{d}y,\end{aligned}$$

where $A = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, 0 < y < 1, xy \leqslant z\}$ (see Figure 3).

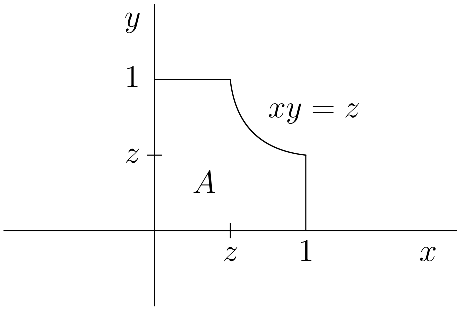

Figure 3: Event A in the xy–plane

Then,

$$\begin{aligned}\Pr(Z \leqslant z) &= \iint_A \mathrm{d}x\mathrm{d}y \\ &= \int_0^z \int_0^1 \mathrm{d}y\mathrm{d}x + \int_z^1 \int_0^{z/x} \mathrm{d}y\mathrm{d}x \\ &= z + \int_z^1 \frac{z}{x}\, \mathrm{d}x \\ &= z - z\ln z,\end{aligned}$$

for $0 < z < 1$. It then follows that the cdf of Z is

$$F_Z(z) = \begin{cases} 0 & \text{if } z \leqslant 0, \\ z - z\ln z & \text{if } 0 < z < 1, \\ 1 & \text{if } z \geqslant 1, \end{cases}$$

and the corresponding pdf of Z is $f_Z(z) = \begin{cases} -\ln z & \text{if } 0 < z < 1, \\ 0 & \text{otherwise.} \end{cases}$ □

5. *[Exercise 11 on page 136 in the text]*

   Suppose that two persons make an appointment to meet between 5 PM and 6 PM at a certain location and they agree that neither person will wait more than 10 minutes for each person. If they arrive independently at random times between 5 PM and 6 PM, what is the probability that they will meet?

   ***Solution**:* Let X denote the arrival time of the first person and Y that of the second person. Then X and Y are independent and uniformly distributed on the interval (5 PM, 6 PM), in hours. It then follows that the joint pdf of X and Y is

   $$f_{(X,Y)}(x,y) = \begin{cases} 1 & \text{if } 5 \text{ PM} < x < 6 \text{ PM}, 5 \text{ PM} < x < 6 \text{ PM}, \\ 0 & \text{otherwise.} \end{cases}$$

   Define $W = |X - Y|$; this is the time that one person would have to wait for the other one. Then, W takes on values, w, between 0 and 1 (in hours). The probability that that a person would have to wait more than 10 minutes is

   $$\Pr(W > 1/6),$$

   since the time is being measured in hours. It then follows that the probability that the two persons will meet is

   $$1 - \Pr(W > 1/6) = \Pr(W \leqslant 1/6) = F_W(1/6).$$

   We will therefore first find the cdf of W. To do this, we compute

   $$\begin{aligned} \Pr(W \leqslant w) &= \Pr(|X - Y| \leqslant w), \quad \text{for } 0 < w < 1, \\ &= \iint_A f_{(X,Y)}(x,y)\, \mathrm{d}x\mathrm{d}y, \end{aligned}$$

   where A is the event

   $$A = \{(x,y) \in \mathbb{R}^2 \mid 5\,\text{PM} < x < 6\,\text{PM}, 5\,\text{PM} < y < 6\,\text{PM}, |x-y| \leqslant w\}.$$

   This event is pictured in Figure 4.

   We then have that

   $$\begin{aligned} \Pr(W \leqslant w) &= \iint_A \mathrm{d}x\mathrm{d}y \\ &= \text{area}(A), \end{aligned}$$

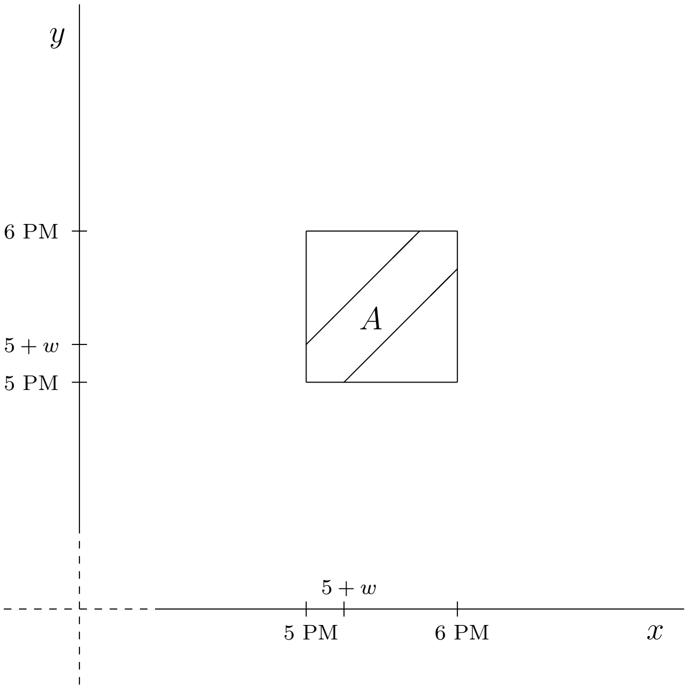

Figure 4: Event A in the xy–plane

where the area of A can be computed by subtracting from 1 the area of the two corder triangles shown in Figure 4:

$$\begin{aligned} \Pr(W \leqslant w) &= 1-(1-w)^2 \\ &= 2w-w^2. \end{aligned}$$

Consequently, $F_W(w) = 2w - w^2$ for $0 < w < 1$. Thus the probability that the two persons will meet is

$$F_W(1/6) = 2\cdot\frac{1}{6} - \left(\frac{1}{6}\right)^2 = \frac{11}{36},$$

or about 30.56%. $\square$