Solutions to Assignment #16

1. Suppose X and Y are independent and let $g_1(X)$ and $g_2(Y)$ be functions for which $E(g_1(X)g_2(Y))$ exists. Show that

$$E(g_1(X)g_2(Y)) = E(g_1(X)) \cdot E(g_2(Y))$$

Conclude therefore that if X and Y are independent and $E(|XY|)$ is finite, then

$$E(XY) = E(X) \cdot E(Y).$$

***Solution**:* We preset the proof for the case in which both X and Y are continuous random variables. The case in which both are discrete is similar.

Since X and Y are independent, the joint pdf of X and Y is

$$f_{(X,Y)}(x,y) = f_X(x) \cdot f_Y(y) \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Then,

$$\begin{aligned}
E(g_1(X)g_2(Y)) &= \iint_{\mathbb{R}^2} g_1(x)g_2(y)f_{(X,Y)}(x,y)\ \mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x)g_2(y)f_X(x)f_Y(y)\ \mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty} g_2(y)f_Y(y) \int_{-\infty}^{\infty} g_1(x)f_X(x)\ \mathrm{d}x\ \mathrm{d}y \\
&= \int_{-\infty}^{\infty} g_2(y)f_Y(y)\ E(g_1(X))\ \mathrm{d}y \\
&= E(g_1(X)) \cdot \int_{-\infty}^{\infty} g_2(y)f_Y(y)\ \mathrm{d}y \\
&= E(g_1(X)) \cdot E(g_2(Y)),
\end{aligned}$$

which was to be shown.

For the case in which $g_1(x) = x$ and $g_2(y) = y$, we obtain

$$E(XY) = E(X) \cdot E(Y),$$

provided that X and Y are independent. □

2. Suppose X and Y are independent random variables for which the moment generating functions exist on some common interval of values of t. Show that

$$\psi_{X+Y}(t) = \psi_X(t)\cdot\psi_Y(t)$$

for t is the given interval.

***Solution**:* Observe that $e^{t(x+y)} = e^{tx}e^{ty}$ for all $x,\ y,\ t\in\mathbb{R}$. Thus, applying the result of the previous problem to $g_1(x) = e^{tx}$ and $g_2(y) = e^{ty}$, we obtain

$$\begin{array}{rcl}\psi_{(X,Y)}(t) & = & E(e^{t(X+Y)}) \\ & = & E(e^{tX}e^{tY}) \\ & = & E(e^{tX})\cdot E(e^{tY}),\end{array}$$

since X and Y are independent. It then follows that

$$\psi_{X+Y}(t) = \psi_X(t)\cdot\psi_Y(t),$$

for t in the given interval. □

3. Suppose that $X \sim \text{Normal}(\mu,\sigma^2)$ and define $Y = \dfrac{X-\mu}{\sigma}$.

Prove that $Y \sim \text{Normal}(0,1)$

***Solution**:* Since $X \sim \text{Normal}(\mu,\sigma^2)$, its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-(x-\mu)^2/2\sigma^2} \quad \text{for } -\infty < x < \infty.$$

We compute the cdf of Y:

$$\begin{array}{rcl}F_Y(y) & = & \Pr(Y\leqslant y), \qquad \text{for } -\infty < y < \infty, \\ & = & \Pr\left(\dfrac{X-\mu}{\sigma}\leqslant y\right) \\ & = & \Pr\left(X\leqslant \sigma y+\mu\right) \\ & = & F_X(\sigma y+\mu).\end{array}$$

Differentiating with respect to y we then get that

$$
\begin{aligned}
f_{Y}(y) &= F'_{X}(\sigma y + \mu) \cdot \sigma, \qquad \text{for } -\infty < y < \infty, \\
&= \sigma\ f_{X}(\sigma y + \mu) \\
&= \sigma \cdot \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-(\sigma y + \mu - \mu)^2/2\sigma^2} \\
&= \frac{1}{\sqrt{2\pi}}\ e^{-y^2/2},
\end{aligned}
$$

for $-\infty < y < \infty$, which is the pdf of a Normal$(0, 1)$ distribution. □

4. Let X_1 and X_2 denote independent, Normal$(0, \sigma^2)$ random variables, where $\sigma > 0$. Define the random variables

$$
\overline{X} = \frac{X_1 + X_2}{2} \quad \text{and} \quad Y = \frac{(X_1 - X_2)^2}{2\sigma^2}.
$$

Determine the distributions of $\overline{X}$ and Y.

Suggestion: To obtain the distribution for Y, first show that

$$
\frac{X_1 - X_2}{\sqrt{2}\ \sigma} \sim \text{Normal}(0, 1).
$$

***Solution**:* Since X_1 and X_2 are Normal$(0, \sigma^2)$ distribution, they both have the mgf

$$
\psi(t) = e^{\sigma^2 t^2/2} \quad \text{for } -\infty < t < \infty.
$$

It then follows that the mgf of $\dfrac{X_1}{2}$ is

$$
\begin{aligned}
\psi_{X_1/2}(t) &= E(e^{tX_1/2}) \\
&= E(e^{(t/2)X_1}) \\
&= \psi(t/2) \\
&= e^{\sigma^2 t^2/8} \quad \text{for } -\infty < t < \infty.
\end{aligned}
$$

Similarly,

$$
\psi_{X_2/2}(t) = e^{\sigma^2 t^2/8} \quad \text{for } -\infty < t < \infty.
$$

It then follows from problem (2) in this assignment, that the mgf of $\overline{X}$ is

$$\begin{aligned}
\psi_{\overline{X}}(t) &= \psi_{X_1/2+X_2/2}(t) \\
&= \psi_{X_1/2}(t) \cdot \psi_{X_2/2}(t) \\
&= e^{\sigma^2 t^2/8} \cdot e^{\sigma^2 t^2/8} \\
&= e^{\sigma^2 t^2/4} \\
&= e^{(\sigma^2/2)t^2/2}, \quad \text{for } -\infty < t < \infty,
\end{aligned}$$

which is the mgf of a Normal$(0, \sigma^2/2)$ distribution. It then follows that $\overline{X} \sim$ Normal$(0, \sigma^2/2)$ and, therefore, its pdf is

$$\begin{aligned}
f_{\overline{X}}(x) &= \frac{1}{\sqrt{2\pi}\ (\sigma/\sqrt{2})}\ e^{-x^2/\sigma^2} \\
&= \frac{1}{\sqrt{\pi}\ \sigma}\ e^{-x^2/\sigma^2} \quad \text{for } -\infty < x < \infty.
\end{aligned}$$

Next, let $Z = \dfrac{X_1 - X_2}{\sqrt{2}\ \sigma}$, so that $Y = Z^2$. We show that $Z \sim$ Normal$(0, 1)$. To see why this is so, compute the mgf of Z:

$$\begin{aligned}
\psi_Z(t) &= \psi_{X_1/(\sqrt{2}\sigma)+X_2/(\sqrt{2}\sigma)}(t) \\
&= \psi_{X_1/(\sqrt{2}\sigma)}(t) \cdot \psi_{X_2/(\sqrt{2}\sigma)}(t) \\
&= \psi(t/\sqrt{2}\sigma) \cdot \psi(t/\sqrt{2}\sigma) \\
&= [\psi(t/\sqrt{2}\sigma)]^2 \\
&= [e^{\sigma^2(t^2/2\sigma^2)/2}]^2 \\
&= e^{t^2/2}, \quad \text{for } -\infty < t < \infty,
\end{aligned}$$

which is the mgf of a Normal$(0, 1)$ distribution. Thus, Z has a Normal$(0, 1)$ distribution.

It then follows that $Y = Z^2 \sim \chi^2(1)$. Thus, the pdf of Y is

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \cdot \dfrac{1}{\sqrt{y}}\, e^{-y/2} & \text{if } y > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

□

5. Let X_1, X_2, $\overline{X}$ and Y be as in the previous problem. Prove that $\overline{X}$ and Y are independent.

Solution*:* In the previous problem we saw that $\overline{X} \sim \text{Normal}(0, \sigma^2/2)$ and $Y \sim \chi^2(1)$. It then follows that the pdf of $\overline{X}$ is

$$f_{\overline{X}}(x) = \frac{1}{\sqrt{\pi}\,\sigma}\, e^{-x^2/\sigma^2} \quad \text{for } -\infty < x < \infty,$$

and that of Y is

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \cdot \dfrac{1}{\sqrt{y}}\, e^{-y/2} & \text{if } y > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

To show that $\overline{X}$ and Y are independent, we need to show that the joint pdf of $\overline{X}$ and Y is the product of their marginal pdfs. To see why this is so, first compute the joint cdf:

$$\begin{aligned} F_{(\overline{X},Y)}(x,y) &= \Pr(\overline{X} \leqslant x, Y \leqslant y), \quad \text{for } -\infty < x < \infty,\ y > 0, \\ &= \Pr\left(\frac{X_1 + X_2}{2} \leqslant x, \frac{(X_1 - X_2)^2}{2\sigma^2} \leqslant y\right) \\ &= \Pr\left(X_1 + X_2 \leqslant 2x, |X_2 - X_1| \leqslant \sigma\sqrt{2y}\right) \\ &= \iint_A f_{(X_1,X_2)}(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2, \end{aligned}$$

where

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 \leqslant 2x, x_1 - \sigma\sqrt{2y} \leqslant x_2 \leqslant x_1 + \sigma\sqrt{2y}\}$$

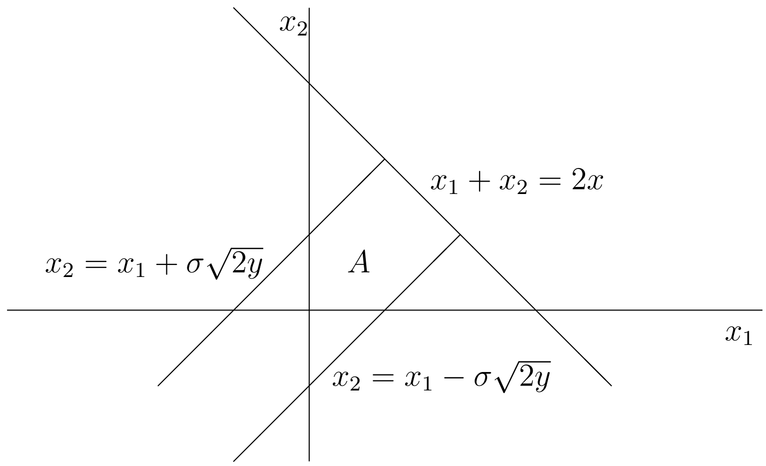

Figure 1: Event A in the x_1x_2–plane

is pictured in Figure 1.

Now, since X_1 and X_2 are independent, it follows that their joint pdf is

$$\begin{aligned} f_{(X_1,X_2)}(x_1,x_2) &= f_{X_1}(x_1)\cdot f_{X_2}(x_2) \\ &= \frac{1}{\sqrt{2\pi}\ \sigma}e^{-x_1^2/2\sigma^2}\cdot\frac{1}{\sqrt{2\pi}\ \sigma}e^{-x_2^2/2\sigma^2} \\ &= \frac{1}{2\pi\sigma^2}\ e^{-(x_1^2+x_2^2)/2\sigma^2}, \end{aligned}$$

for $(x_1,x_2)\in\mathbb{R}^2$. We then have that

$$F_{(\overline{X},Y)}(x,y) = \frac{1}{2\pi\sigma^2}\iint_A e^{-(x_1^2+x_2^2)/2\sigma^2}\ \mathrm{d}x_1\mathrm{d}x_2.$$

To evaluate this integral, we make the change of variable

$$\begin{cases} u = x_1 + x_2, \\ v = x_2 - x_1. \end{cases}$$

This change of variables turns the region A in the x_1x_2–plane into the region:

$$A^* = \{(u,v)\in\mathbb{R}^2 \mid -\infty < u \leqslant 2x, -\sigma\sqrt{2y}\leqslant v\leqslant \sigma\sqrt{2y}\}$$

in the uv–plane.

Solving for x_1 and x_2 in terms of u and v, we obtain

$$\begin{cases} x_1 = \frac{1}{2}u - \frac{1}{2}v, \\ \\ v = \frac{1}{2}u + \frac{1}{2}v, \end{cases}$$

from which we get that

$$x_1^2 + x_2^2 = \frac{1}{2}(u^2 + v^2).$$

Thus, by the Change of Variables Theorem in Multivariable Calculus,

$$F_{(\overline{X},Y)}(x,y) = \frac{1}{2\pi\sigma^2} \iint_{A^*} e^{-(u^2+v^2)/4\sigma^2} \left| \frac{\partial(x_1,x_2)}{\partial(u,v)} \right| \, \mathrm{d}u\mathrm{d}v,$$

where $\dfrac{\partial(x_1,x_2)}{\partial(u,v)}$ denotes the Jacobian of the change of variables:

$$\frac{\partial(x_1,x_2)}{\partial(u,v)} = \det \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}.$$

We then have that Calculus,

$$\begin{aligned} F_{(\overline{X},Y)}(x,y) &= \frac{1}{4\pi\sigma^2} \iint_{A^*} e^{-(u^2+v^2)/4\sigma^2} \, \mathrm{d}u\mathrm{d}v \\ &= \frac{1}{4\pi\sigma^2} \int_{-\infty}^{2x} \int_{-\sigma\sqrt{2y}}^{\sigma\sqrt{2y}} e^{-u^2/4\sigma^2} \cdot e^{-v^2/4\sigma^2} \, \mathrm{d}v\mathrm{d}u \\ &= \frac{1}{4\pi\sigma^2} \int_{-\infty}^{2x} e^{-u^2/4\sigma^2} 2 \cdot \int_0^{\sigma\sqrt{2y}} e^{-v^2/4\sigma^2} \, \mathrm{d}v \, \mathrm{d}u, \end{aligned}$$

where we have used the fact that $e^{-v^2/4\sigma^2}$ is an even function of v. We then have that

$$F_{(\overline{X},Y)}(x,y) = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{2x} e^{-u^2/4\sigma^2} \, \mathrm{d}u \cdot \int_0^{\sigma\sqrt{2y}} e^{-v^2/4\sigma^2} \, \mathrm{d}v.$$

Next, use the Fundamental Theorem of Calculus and the Chain Rule to differentiate with respect to x and y to obtain that

$$f_{(\overline{X},Y)}(x,y) = \frac{1}{2\pi\sigma^2} \cdot 2e^{-(2x)^2/4\sigma^2} \cdot e^{-(\sigma\sqrt{2y})^2/4\sigma^2} \cdot \sigma \frac{2}{2\sqrt{2y}}.$$

Simplifying we obtain

$$f_{(\overline{X},Y)}(x,y) = \frac{1}{\pi\sigma} \cdot e^{-x^2/\sigma^2} \cdot e^{-y/2} \cdot \frac{1}{\sqrt{2y}},$$

and, after rearranging terms,

$$f_{(\overline{X},Y)}(x,y) = \frac{1}{\sqrt{\pi}\sigma}\ e^{-x^2/\sigma^2} \cdot \frac{1}{\sqrt{2\pi}}\ \frac{1}{\sqrt{y}}\ e^{-y/2},$$

for $-\infty < x < \infty$ and $y > 0$.

We therefore conclude that

$$f_{(\overline{X},Y)}(x,y) = f_{\overline{X}}(x) \cdot f_Y(y)$$

for $-\infty < x < \infty$ and $y > 0$. Hence, $\overline{X}$ and Y are independent. □