Solutions to Assignment #17

1. We have seen in the lecture that if X has a Poisson distribution with parameter $\lambda > 0$, then it has the pmf:

$$p_{_X}(k) = \frac{\lambda^k}{k!}\, e^{-\lambda} \quad \text{for } k = 0, 1, 2, 3, \ldots; \text{ zero elsewhere.}$$

Use the fact that the power series $\displaystyle\sum_{m=0}^{\infty} \frac{x^m}{m!}$ converges to e^x for all real values of x to compute the mgf of X.

Use the mgf of X to determine the mean and variance of X.

***Solution**:* Compute

$$\begin{aligned} \psi_{_X}(t) &= E(e^{tX}) \\ &= \sum_{k=0}^{\infty} e^{tk} p_{_X}(k) \\ &= \sum_{k=0}^{\infty} (e^t)^k \frac{\lambda^k}{k!}\, e^{-\lambda} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} \\ &= e^{-\lambda}\, e^{\lambda e^t} \\ &= e^{\lambda(e^t - 1)}, \end{aligned}$$

for all $t \in \mathbb{R}$.

Differentiating the mgf we obtain

$$\psi'_{_X}(t) = \lambda e^t\, e^{\lambda(e^t-1)},$$

and

$$\psi''_{_X}(t) = \lambda e^t\, e^{\lambda(e^t-1)} + \lambda^2 e^{2t}\, e^{\lambda(e^t-1)}.$$

We then get that the expected value of X is

$$E(X) = \psi'_{_X}(0) = \lambda,$$

and the second moment of X is

$$E(X^2) = \psi_X''(0) = \lambda + \lambda^2.$$

Consequently, the variance of X is

$$\text{var}(X) = E(X^2) - \lambda^2 = \lambda.$$

□

2. Let $X_1,\ X_2,\ \ldots\ X_m$ be independent random variables satisfying $X_i \sim \text{Poisson}(\lambda)$ for all $i = 1, 2, \ldots, m$ and some $\lambda > 0$. Define

$$Y = X_1 + X_2 + \cdots + X_m.$$

Determine the distribution of Y; that is, compute its pmf.

***Solution**:* Use the independence of the random variables X_i, for $i = 1, 2, \ldots, m$, to compute the moment generating function of Y:

$$\begin{aligned} \psi_Y(t) &= \psi_{X_1+X_2+\cdots+X_m}(t) \\ &= \psi_{X_1}(t)\cdot\psi_{X_2}(t)\cdots\psi_{X_m}(t) \\ &= e^{\lambda(e^t-1)}\cdot e^{\lambda(e^t-1)}\cdots e^{\lambda(e^t-1)} \\ &= e^{m\lambda(e^t-1)}, \end{aligned}$$

which is the mgf of a Poisson$(m\lambda)$ random variable. It then follows that $Y \sim \text{Poisson}(m\lambda)$ and, therefore, its pmf is given by

$$p_Y(k) = \frac{(m\lambda)^k}{k!}\, e^{-m\lambda} \quad \text{for } k = 0, 1, 2, 3, \ldots; \text{ zero elsewhere.}$$

□

3. *[Exercise 2 on page 262 in the text]*

   Suppose that on a given weekend the number of accidents at a certain intersection has a Poisson distribution with mean 0.7. What is the probability that there will be at least three accidents in the intersection during the weekend?

***Solution**:* Let X denote the number of accidents that occur in the intersection during a weekend. Then, we are assuming that $X \sim \text{Poisson}(0.7)$ and therefore

$$\Pr(X = k) = \frac{(0.7)^k}{k!}\, e^{-0.7} \quad \text{for } k = 0, 1, 2, 3, \ldots; \text{ zero elsewhere.}$$

Thus, the probability of at least three accidents during that weekend, $\Pr(X \geqslant 3)$, is

$$\begin{aligned} \Pr(X \geqslant 3) &= 1 - \Pr(X < 3) \\ &= 1 - \Pr(X = 0) - \Pr(X = 1) - \Pr(X = 2) \\ &= 1 - e^{-0.7} - (0.7)e^{-0.7} - \frac{(0.7)^2}{2} e^{-0.7}, \end{aligned}$$

or about 3.4%. □

4. *[Exercise 6 on page 262 in the text]*

Suppose that a certain type of magnetic tape contains, on average, three defects per 1000 feet. What is the probability that a roll of tape 1200 feet long contains no defects?

***Solution**:* Let X denote the number of defects in one foot of tape. Then $X \sim \text{Poisson}(\lambda)$, where

$$\lambda = \frac{3}{1000} = 0.003.$$

It is reasonable to assume that the number of defects in a foot of the tape is independent of the number of defects in any other foot of the tape. Thus, if we let Y denote the number of defects in 1200 feet of the tape, it follows from problem 2 in this assignment that

$$Y \sim \text{Poisson}(1200\lambda) \quad \text{or} \quad \text{Poisson}(3.6).$$

Thus,

$$\Pr(Y = k) = \frac{(3.6)^k}{k!}\, e^{-3.6} \quad \text{for } k = 0, 1, 2, 3, \ldots; \text{ zero elsewhere.}$$

Consequently, the probability of no defects in the 1200 feet of tape is

$$\Pr(Y = 0) = e^{-3.6}$$

or about 2.73%. □

5. *[Exercise 8 on page 262 in the text]*

   Suppose that X_1 and X_2 are independent random variables and that X_i has a Poisson distribution with mean λ_i ($i = 1, 2$). For a fixed value of k ($k = 0, 1, 2, 3, \ldots$), determine the conditional distribution of X_1 given that $X_1 + X_2 = k$.

   Solution: Let $Y = X_1 + X_2$. Then, since X_1 and X_2 are independent with $X_i \sim \text{Poisson}(\lambda_i)$, for $i = 1, 2$, $Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$. Thus, the pmf of Y is

$$\Pr(Y = k) = \frac{(\lambda_1 + \lambda_2)^k}{k!}\, e^{-\lambda_1 - \lambda_2} \quad \text{for } k = 0, 1, 2, 3, \ldots; \text{ zero elsewhere.}$$

   We compute the conditional probability $\Pr(X_1 = m \mid Y = k)$ for $m = 0, 1, 2, 3, \ldots$

$$\begin{aligned} \Pr(X_1 = m \mid Y = k) &= \frac{\Pr(X_1 = m, Y = k)}{\Pr(Y = k)} \\ &= \frac{\Pr(X_1 = m, X_1 + X_2 = k)}{\Pr(Y = k)} \\ &= \frac{\Pr(X_1 = m, X_2 = k - m)}{\Pr(Y = k)} \\ &= \frac{\Pr(X_1 = m) \cdot \Pr(X_2 = k - m)}{\Pr(Y = k)}, \end{aligned}$$

   by the independence of X_1 and X_2; so that, for $m = 0, 1, 2, \ldots, k$,

$$\begin{aligned} \Pr(X_1 = m \mid Y = k) &= \frac{\dfrac{\lambda_1^m}{m!}\, e^{-\lambda_1} \cdot \dfrac{\lambda_2^{k-m}}{(k-m)!}\, e^{-\lambda_2}}{\dfrac{(\lambda_1 + \lambda_2)^k}{k!}\, e^{-\lambda_1 - \lambda_2}} \\ &= \frac{\dfrac{\lambda_1^m}{m!} \cdot \dfrac{\lambda_2^{k-m}}{(k-m)!}}{\dfrac{(\lambda_1 + \lambda_2)^k}{k!}} \\ &= \frac{k!}{m!(k-m)!} \cdot \frac{\lambda_1^m \lambda_2^{k-m}}{(\lambda_1 + \lambda_2)^k}. \end{aligned}$$

Observing that the first factor in the last expression is the binomial coefficient $\binom{k}{m}$, we can write

$$\begin{aligned}
\Pr(X_1 = m \mid Y = k) &= \binom{k}{m} \cdot \frac{\lambda_1^m}{(\lambda_1+\lambda_2)^m} \cdot \frac{\lambda_2^{k-m}}{(\lambda_1+\lambda_2)^{k-m}} \\
&= \binom{k}{m} \cdot \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^m \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{k-m} \\
&= \binom{k}{m} p^m (1-p)^{k-m},
\end{aligned}$$

for $m = 0, 1, 2, \ldots, k$, where $p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$.

Hence, the conditional distribution of X_1 given $X_1 + X_2 = k$ is Binomial(p, k) with $p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$. □