## Solutions to Assignment #19

1. Prove that if $X$ and $Y$ are independent random variables,

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y).$$

Generalize this result to $n$ independent random variables $X_1, X_2, \ldots, X_n$.

***Solution**:* Let $\mu_X = E(X)$, $\mu_Y = E(Y)$, and compute

$$\begin{aligned}\text{var}(X+Y) &= E[(X+Y-(\mu_X+\mu_Y))^2] \\ &= E[((X-\mu_X)+(Y-\mu_Y))^2] \\ &= E[(X-\mu_X)^2 + 2(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2] \\ &= E[(X-\mu_X)^2] + 2E[(X-\mu_X)(Y-\mu_Y)] + E[(Y-\mu_Y)^2] \\ &= \text{var}(X) + \text{var}(Y) + 2E[(X-\mu_X)(Y-\mu_Y)].\end{aligned}$$

Thus, since $X$ and $Y$ are independent,

$$\begin{aligned}\text{var}(X+Y) &= \text{var}(X) + \text{var}(Y) + 2E[(X-\mu_X)]E[(Y-\mu_Y)] \\ &= \text{var}(X) + \text{var}(Y) + 2(E(X)-\mu_X)(E(Y)-\mu_Y) \\ &= \text{var}(X) + \text{var}(Y).\end{aligned}$$

An induction argument can now be used to show that if $X_1, X_2, \ldots, X_n$ are independent random variables, then

$$\text{var}\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} \text{var}(X_k).$$

□

2. Let $X_n \sim \text{Poisson}(n)$, for $n = 1, 2, 3, \ldots$, and define $Z_n = \dfrac{X_n - n}{\sqrt{n}}$ for $n = 1, 2, 3, \ldots$ Use the mgf Convergence Theorem to find the limiting distribution of $Z_n$.

***Solution**:* We first compute the mgf of $Z_n$:

$$\begin{aligned}\psi_{Z_n}(t) &= E\left(e^{\frac{t}{\sqrt{n}}X_n - t\sqrt{n}}\right) \\ &= e^{-t\sqrt{n}}\psi_{X_n}\left(\frac{t}{\sqrt{n}}\right),\end{aligned}$$

where $\psi_{X_n}(t) = e^{n(e^t-1)}$ for all $t \in \mathbb{R}$. Thus,

$$\psi_{Z_n}(t) \;=\; e^{-t\sqrt{n}}e^{n(e^{t/\sqrt{n}}-1)},$$

for all $t \in \mathbb{R}$.

Next, use the fact that

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

to obtain that

$$e^{t/\sqrt{n}} = 1 + \frac{t}{\sqrt{n}} + \frac{1}{2}\,\frac{t^2}{n} + \frac{1}{3!}\,\frac{t^3}{n\sqrt{n}} + \frac{1}{4!}\,\frac{t^4}{n^2} + \cdots$$

Then,

$$e^{t/\sqrt{n}} - 1 = \frac{t}{\sqrt{n}} + \frac{1}{2}\,\frac{t^2}{n} + \frac{1}{3!}\,\frac{t^3}{n\sqrt{n}} + \frac{1}{4!}\,\frac{t^4}{n^2} + \cdots,$$

and

$$n(e^{t/\sqrt{n}} - 1) = t\sqrt{n} + \frac{t^2}{2} + \frac{1}{3!}\,\frac{t^3}{\sqrt{n}} + \frac{1}{4!}\,\frac{t^4}{n} + \cdots$$

Exponentiating we then have that

$$e^{n(e^{t/\sqrt{n}}-1)} = e^{t\sqrt{n}} \cdot e^{t^2/2} \cdot e^{\frac{1}{3!}\,\frac{t^3}{\sqrt{n}} + \frac{1}{4!}\,\frac{t^4}{n} + \cdots},$$

or

$$\psi_{Z_n}(t) = e^{-t\sqrt{n}}e^{n(e^{t/\sqrt{n}}-1)} = e^{t^2/2} \cdot e^{\frac{1}{3!}\,\frac{t^3}{\sqrt{n}} + \frac{1}{4!}\,\frac{t^4}{n} + \cdots}.$$

Since the exponent in the last exponential tends to 0 as $n \to \infty$, we get that

$$\lim_{n\to\infty} \psi_{Z_n}(t) = e^{t^2/2},$$

which is the mgf of $Z \sim \text{Normal}(0, 1)$. Hence, by the mgf Convergence Theorem, $Z_n$ converges in distribution to a Normal$(0, 1)$ random variable. □

3. Let $X$ and $Y$ be independent continuous random variables with pdfs $f_X$ and $f_Y$, respectively. Let $Z = X + Y$ and show that the pdf for $Z$ is given by

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(u) f_Y(z-u)\,\mathrm{d}u$$

for all $z \in \mathbb{R}$. This is known as the **convolution** of $f_X$ and $f_Y$.

*Suggestion:* To evaluate the double integral defining $P(X + Y \leq z)$, make the change of variables $u = x$ and $v = x + y$. Observe that with this change of variables, the region of integration in the $uv$–plane becomes: $\{(u, v) \in \mathbb{R}^2 \mid -\infty < u < \infty, -\infty < v < z\}$.

***Solution:*** Suppose $X$ and $Y$ are independent continuous random variables with pdf's $f_X$ and $f_Y$, respectively. Then the joint distribution of $X$ and $Y$ is

$$f_{(X,Y)}(x, y) = f_X(x) f_Y(y) \quad \text{for } -\infty < x < \infty, \; -\infty < y < \infty.$$

Put $Z = X + Y$, and compute

$$\Pr(Z \leqslant z) = \Pr(X + Y \leqslant z) = \iint_A f_X(x) f_Y(y) \, \mathrm{d}x\mathrm{d}y,$$

where

$$A = \{(x, y) \in \mathbb{R}^2 | x + y \leqslant z\}.$$

Make the change of variables $u = x$, $v = x + y$. Then, the region $A$ is transformed onto the region

$$A^* = \Big\{(u, v) \in \mathbb{R}^2 | -\infty < u < \infty, -\infty < v < z\Big\}$$

in the uv-plane.
By the Change of Variables Theorem,

$$\Pr(Z \leqslant z) = \iint_{A^*} f_X(u) f_Y(v - u) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \mathrm{d}u\mathrm{d}v,$$

where

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = 1$$

We then have that

$$\Pr(Z \leqslant z) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_X(u) f_Y(v - u) \, \mathrm{d}u\mathrm{d}v,$$

and therefore the cdf of $Z$ is

$$F_Z(z) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_X(u) f_Y(v - u) \, \mathrm{d}u\mathrm{d}v.$$

Thus, by the Fundamental Theorem of Calculus,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(u) f_Y(z-u)\, \mathrm{d}u.$$

□

4. Let $X$ and $Y$ be independent $\chi^2(1)$ random variables. Recall that this means that $X$ and $Y$ both have the pdf

$$f(u) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}\,\dfrac{1}{\sqrt{u}}\, e^{-u/2} & \text{if } u > 0, \\ \\ 0 & \text{elsewhere.} \end{cases}$$

Let $Z = X+Y$ and use the convolution formula derived in the previous problem to compute the pdf of $Z$.

(*Hint:* The distribution of $Z$ is a familiar one).

***Solution:*** Let $X, Y \sim \chi^2(1)$ be independent random variables. Then,

$$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{x}} e^{-x/2} & x > 0, \\ 0 & \text{elsewhere,} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-y/2} & y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Put $Z = X + Y$. We use the convolution formula derived in the previous problem to compute $f_Z(z)$ for $z > 0$:

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(u) f_Y(z-u)\, \mathrm{d}u \\ &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{u}} e^{-u/2} f_Y(z-u)\, \mathrm{d}u, \end{aligned}$$

since $f_X(u)$ is zero for negative values of $u$. Similarly, since $f_Y(z-u) = 0$ for $z - u < 0$, we get that

$$\begin{aligned} f_Z(z) &= \int_0^z \frac{1}{\sqrt{2\pi}\sqrt{u}} e^{-u/2} \frac{1}{\sqrt{2\pi}\sqrt{z-u}} e^{-(z-u)/2}\, \mathrm{d}u \\ &= \frac{e^{-z/2}}{2\pi} \int_0^z \frac{1}{\sqrt{u}\sqrt{z-u}}\, \mathrm{d}u. \end{aligned}$$

Next, make the change of variables $w = \dfrac{u}{z}$. Then, $\mathrm{d}u = z\mathrm{d}w$ and

$$\begin{aligned} f_Z(z) &= \frac{e^{-z/2}}{2\pi}\int_0^1 \frac{z}{\sqrt{zw}\sqrt{z-zw}}\,\mathrm{d}w \\ &= \frac{e^{-z/2}}{2\pi}\int_0^1 \frac{1}{\sqrt{w}\sqrt{1-w}}\,\mathrm{d}w. \end{aligned}$$

Making the change of variables $s = \sqrt{w}$, we get that $w = s^2$ and $\mathrm{d}w = 2s\mathrm{d}s$, so that

$$\begin{aligned} f_Z(z) &= \frac{e^{-z/2}}{\pi}\int_0^1 \frac{1}{\sqrt{1-s^2}}\,\mathrm{d}s \\ &= \frac{e^{-z/2}}{\pi}\left[\arcsin(s)\right]_0^1 \\ &= \frac{1}{2}\,e^{-z/2} \quad \text{for } z > 0, \end{aligned}$$

and zero otherwise. It then follows that $Z = X + Y$ has the pdf of an Exponential(2) random variable. □

5. Use the result of the previous problem to compute the moment generating function of a $\chi^2(1)$ random variable.

   ***Solution***: We have seen that if $X$ and $Y$ are independent $\chi^2(1)$ distribution, then their sum is an Exponential(2) distribution. Consequently,

   $$\psi_{X+Y}(t) = \psi_X(t)\cdot\psi_Y(t) = \left[\psi_X(t)\right]^2,$$

   since $X$ and $Y$ have the same distribution. We then get that

   $$\left[\psi_X(t)\right]^2 = \frac{1}{1-2t} \quad \text{for} \quad t < \frac{1}{2}.$$

   Thus the mgf of $X \sim \chi^2(1)$ is

   $$\psi_X(t) = \frac{1}{\sqrt{1-2t}} \quad \text{for} \quad t < \frac{1}{2}.$$

   □