Solutions to Assignment #1

1. Let $\mathcal{C}$ denote a sample space and A be a subset of $\mathcal{C}$. Establish the following set theoretic identities:

   (a) $A \cap \emptyset = \emptyset$,

   (b) $A \cup \emptyset = A$;

   where $\emptyset$ denotes the empty set. Justify your steps.

   Solution:

   (a) *Proof:* The set of elements that are common to both A and $\emptyset$ is the empty set since $\emptyset$ has no elements. It then follows that

   $$A \cap \emptyset = \emptyset. \qquad \square$$

   (b) *Proof:* $x \in A \cup \emptyset$ iff $x \in A$ or $x \in \emptyset$. However, $\emptyset$ has no elements. It then follows that $x \in A$. We then have that

   $$A \cup \emptyset \subseteq A.$$

   On the other hand,

   $$A \subseteq A \cup \emptyset.$$

   Hence, $A \cup \emptyset = A$.

   $\square$

2. Let $\mathcal{C}$ denote a sample space and A and B denote subsets of $\mathcal{C}$. Establish the following set theoretic identities:

   (a) $(A^c)^c = A$,

   (b) $(A \cup B)^c = A^c \cap B^c$;

   where A^c denote the complement of A.

   Solution:

   (a) *Proof:*

   $$\begin{aligned} x \in (A^c)^c \quad & \text{iff} \quad x \notin A^c \\ & \text{iff} \quad \text{it is not true that } x \in A^c \\ & \text{iff} \quad \text{it is the case that } x \in A \\ & \text{iff} \quad x \in A \end{aligned}$$

   Hence, $(A^c)^c = A$. $\square$

(b) *Proof:*

$$\begin{aligned} x \in (A \cup B)^c \quad & \text{iff} \quad x \notin A \cup B \\ & \text{iff} \quad x \notin A \text{ and } \ x \notin B \\ & \text{iff} \quad x \in A^c \text{ and } \ x \in B^c \\ & \text{iff} \quad x \in A^c \cap B^c \end{aligned}$$

Hence, $(A \cup B)^c = A^c \cap B^c$.

□

3. Let $\mathcal{C}$ denote a sample space and A, B and C denote subsets of $\mathcal{C}$. Prove the following distributive properties:

   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   Solution:

   (a) *Proof:*

$$\begin{aligned} x \in A \cap (B \cup C) \quad & \text{iff} \quad x \in A \text{ and } x \in B \cup C \\ & \text{iff} \quad x \in A \text{ and } (x \in B \text{ or } x \in C) \\ & \text{iff} \quad (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ & \text{iff} \quad x \in A \cap B \text{ or } x \in A \cap C \\ & \text{iff} \quad x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

   Consequently, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

   (b) *Proof:*

$$\begin{aligned} x \in A \cup (B \cap C) \quad & \text{iff} \quad x \in A \text{ or } x \in B \cap C \\ & \text{iff} \quad x \in A \text{ or } (x \in B \text{ and } x \in C) \\ & \text{iff} \quad (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\ & \text{iff} \quad x \in A \cup B \text{ and } x \in A \cup C \\ & \text{iff} \quad x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

   Consequently, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

□

4. Let A and B be subsets of the sample space $\mathcal{C}$. The *set difference* $A \backslash B$ is defined to be

$$A \backslash B = \{x \in A \mid x \notin B\};$$

   thus, $A \backslash B$ is a subset of A that contains those elements in A which are not in B.

   Prove that

(a) $A\backslash B = A \cap B^c$,

(b) $B\backslash(A \cap B) = A^c \cap B$

Solution:

(a) *Proof:*

$$\begin{array}{rcl} x \in A\backslash B & \text{iff} & x \in A \text{ and } x \notin B \\ & \text{iff} & x \in A \text{ and } x \in B^c \\ & \text{iff} & x \in A \cap B^c. \end{array}$$

Consequently, $A\backslash B = A \cap B^c$. □

(b) *Proof:* Using the result from part (a), De Morgan's laws and the distributive property, we get

$$\begin{array}{rcl} B\backslash(A \cup B) & = & B \cap (A \cap B)^c \\ & = & B \cap (A^c \cup B^c) \\ & = & (B \cap A^c) \cup (B \cap B^c) \\ & = & (B \cap A^c) \cup \emptyset \\ & = & B \cap A^c, \end{array}$$

where we have also used the result proved in problem (1)(b).

□

5. *[Exercise 1 on page 12 in the text]*

Suppose that $A \subseteq B$. Prove that $B^c \subseteq A^c$.

Proof: If $x \in B^c$, then $x \notin B$. It then follows that $x \notin A$, since A is a subset of B. Thus, $x \in A^c$.

We have therefore shown that

$$x \in B^c \Rightarrow x \in A^c;$$

in other words, $B^c \subseteq A^c$. □