Solutions to Assignment #20

1. Let $X_1, X_2, X_3, \ldots$ denote a sequence of independent, identically distributed random variables with mean μ. Assume that the moment generating function of X_1 exists in some interval around 0. Use the mgf Convergence Theorem to show that the sample means, $\overline{X}_n$, converge in distribution to a limiting distribution with pmf

$$p(x) = \begin{cases} 1 & \text{if } x = \mu; \\ 0 & \text{elsewhere.} \end{cases}$$

Proof: The mgf of the sample mean, $\overline{X}_n$, is given by

$$\psi_{\overline{X}_n}(t) = \left[\psi_{X_1}\left(\frac{t}{n}\right)\right]^n,$$

where $\psi_{X_1}(0) = 1$, and $\psi'_{X_1}(0) = \mu$.

To find the limit of $\psi_{\overline{X}_n}(t)$ as $n \to \infty$, we first consider

$$\lim_{n\to\infty} \ln\left(\psi_{\overline{X}_n}(t)\right) = \lim_{n\to\infty} n \ln\left[\psi_{X_1}\left(\frac{t}{n}\right)\right].$$

Since $\psi_{X_1}(0) = 1$, the limit on the right–hand side can be computed by means of L'Hospital's Rule:

$$\begin{aligned} \lim_{n\to\infty} n \ln\left[\psi_{X_1}\left(\frac{t}{n}\right)\right] &= \lim_{n\to\infty} \frac{\ln\left[\psi_{X_1}\left(\frac{t}{n}\right)\right]}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{\frac{1}{\psi_{X_1}\left(\frac{t}{n}\right)}\psi'_{X_1}\left(\frac{t}{n}\right)\cdot\left(-\frac{t}{n^2}\right)}{-\frac{1}{n^2}}. \end{aligned}$$

We then get that

$$\lim_{n\to\infty} \ln\left(\psi_{\overline{X}_n}(t)\right) = \lim_{n\to\infty} t \cdot \frac{\psi'_{X_1}\left(\frac{t}{n}\right)}{\psi_{X_1}\left(\frac{t}{n}\right)} = t \cdot \frac{\psi'_{X_1}(0)}{\psi_{X_1}(0)} = \mu t.$$

It then follows that $\lim\limits_{n\to\infty} \psi_{\overline{X}_n}(t) = e^{\mu t}$, which is the mgf of $p(x)$. $\square$

2. Let $Y_n \sim \text{Binomial}(p, n)$, for $n = 1, 2, 3, \ldots$, and define $Z_n = \dfrac{Y_n - np}{\sqrt{np(1-p)}}$ for $n = 1, 2, 3, \ldots$ Use the Central Limit Theorem to find the limiting distribution of Z_n.

   Suggestion: Recall that Y_n is the sum of n independent Bernoulli(p) trials.

   ***Solution**:* Observe that $Y_n = \displaystyle\sum_{k=1}^{n} X_k$, where $X_1, X_2, X_3, \ldots$ are independent Bernoulli(p) random variables. Thus, by the Central Limit Theorem,

$$\frac{\frac{1}{n}Y_n - p}{\sqrt{p(1-p)}/\sqrt{n}} \xrightarrow{D} Z \sim \text{Normal}(0,1) \quad \text{as} \quad n \to \infty.$$

   Thus, multiplying the numerator and denominator in the fraction by n,

$$\frac{Y_n - np}{\sqrt{np(1-p)}} \xrightarrow{D} Z \sim \text{Normal}(0,1) \quad \text{as} \quad n \to \infty.$$

   Consequently, the limiting distribution of Z_n is $Z \sim \text{Normal}(0,1)$. □

3. *[Exercise 2 on page 290 in the text]*

   Suppose that 75% of the people in a certain metropolitan area live in the city and 25% of the people live in the suburbs. If 1200 people attending certain concert represent a random sample from the metropolitan area, what is the probability that the number of people from the suburbs attending the concert will be fewer than 270?

   ***Solution**:* Let X denote the number of people in the sample that are from the suburbs. Then, $X \sim \text{Binomial}(p, n)$ where $p = 0.25$ and $n = 1200$. Using the approximation

$$\Pr\left(\frac{X - np}{\sqrt{np(1-p)}} \leqslant z\right) \approx \Pr(Z \leqslant z),$$

   for large n, where $\sim$ Normal(0, 1), we have that

$$\begin{aligned}\Pr(X \leqslant 270) &= \Pr\left(\frac{X - 300}{15} \leqslant \frac{270 - 300}{15}\right) \\ &\approx P(Z \leqslant -2) = 1 - F_z(2) \\ &\approx 0.0227.\end{aligned}$$

Thus, the probability that 270 or fewer people in the sample are from the suburbs is about 2.27%. □

4. *[Exercise 4 on page 290 in the text]*

   Suppose that a random sample of size n is to be taken from a distribution for which the mean is μ and the standard deviation is 3. Use the Central Limit Theorem to determine approximately the smallest value of n for which the following relation will be satisfied:

$$\Pr(|\overline{X}_n - \mu| < 0.3) \geqslant 0.95.$$

***Solution**:* By the Central Limit Theorem,

$$\Pr\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \leqslant z\right) \approx \Pr\left(Z \leqslant z\right),$$

for all $z \in \mathbb{R}$ and large values of n, where $Z \sim \text{Normal}(0,1)$. We then get that, for $z > 0$,

$$\Pr\left(\frac{|\overline{X}_n - \mu|}{\sigma/\sqrt{n}} \leqslant z\right) \approx \Pr\left(|Z| \leqslant z\right),$$

for large values of n, where

$$\Pr(|Z| \leqslant z) = \Pr(-z < Z \leqslant z) = F_Z(z) - F_Z(-z) = 2F_Z(z) - 1.$$

We first find $z > 0$ such that $\Pr(|Z| \leqslant z) \geqslant 0.95$ or $2F_Z(z) - 1 \leqslant 0.95$. This yields $z = 1.96$. With this value of z, we then get that, approximately,

$$\Pr\left(\frac{|\overline{X}_n - \mu|}{\sigma/\sqrt{n}} \leqslant 1.96\right) \geqslant 0.95,$$

or

$$\Pr\left(|\overline{X}_n - \mu| \leqslant 1.96\frac{\sigma}{\sqrt{n}}\right) \geqslant 0.95.$$

We therefore choose n so that

$$1.96\frac{\sigma}{\sqrt{n}} \leqslant 0.3,$$

from which we get that

$$n \geqslant \left(\frac{1.96\sigma}{0.3}\right)^2 \approx 384.16.$$

We therefore take n to be 385. □

5. *[Exercise 12 on page 291 in the text]*

   Let X_n be a random variable having a binomial distribution with parameters n and p_n. Assume that $\lim\limits_{n\to\infty} np_n = \lambda$. Prove that the mgf of X_n converges to the mgf of a Poisson distribution with parameter λ as $n \to \infty$.

   Solution*:* Since $X_n \sim \text{Binomial}(p_n, n)$ for all $n = 1, 2, 3, \ldots$, the mgf of X_n is
   $$\psi_{X_n}(t) = (p_n e^t + 1 - p_n)^n,$$
   for $n = 1, 2, 3, \ldots$ We then have that
   $$\psi_{X_n}(t) = (1 + p_n(e^t - 1))^n,$$
   for $n = 1, 2, 3, \ldots$

   Now, for large values of n, $p_n \approx \dfrac{\lambda}{n}$, since $\lim\limits_{n\to\infty} np_n = \lambda$. We therefore have that
   $$\psi_{X_n}(t) \approx \left(1 + \frac{\lambda}{n}(e^t - 1)\right)^n,$$
   for large values of n. Consequently,
   $$\lim_{n\to\infty} \psi_{X_n}(t) = \lim_{n\to\infty} \left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n = e^{\lambda(e^t - 1)},$$
   which is the mgf of a Poisson(λ) random variable. □