Solutions to Assignment #2

1. Let A, B and C be subsets of a sample space $\mathcal{C}$. Prove the following

   (a) If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

   (b) If $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

   Solution:

   (a) *Proof:* If $x \in A \cup B$, then either $x \in A$ or $x \in B$. If $x \in A$ then $x \in C$, since $A \subseteq C$. Similarly, if $x \in B$ then $x \in C$ since $B \subseteq C$. In either case, $x \in C$. We have therefore shown that

   $$x \in A \cup B \Rightarrow x \in C;$$

   that is, $A \cup B \subseteq C$. □

   (b) *Proof:* If $x \in C$, then $x \in A$ and $x \in B$ since both $c \subseteq A$ and $C \subseteq B$ are assumed to be true. It then follows that $x \in A \cap B$. We have thus shown that

   $$x \in C \Rightarrow x \in A \cap B;$$

   that is, $C \subseteq A \cap B$.

   □

2. Let $\mathcal{C}$ be a sample space and $\mathcal{B}$ be a σ–field of subsets of $\mathcal{C}$. Prove that if $\{E_1, E_2, E_3 \ldots\}$ is a sequence of events in $\mathcal{B}$, then

   $$\bigcap_{k=1}^{\infty} E_k \in \mathcal{B}.$$

   Hint: Use De Morgan's Laws.

   Proof: Let $E_1, E_2, E_3, \ldots$ be a sequence of events in $\mathcal{B}$. Then, $E_1^c, E_2^c, E_3^c, \ldots$ are also in $\mathcal{B}$, and therefore

   $$\bigcup_{k=1}^{\infty} E_k^c \in \mathcal{B},$$

   and consequently,

   $$\left(\bigcup_{k=1}^{\infty} E_k^c\right)^c \in \mathcal{B}.$$

It then follows by De Morgan's laws that

$$\bigcap_{k=1}^{\infty} (E_k^c)^c \in \mathcal{B},$$

or

$$\bigcap_{k=1}^{\infty} E_k \in \mathcal{B}.$$

□

3. Let $\mathcal{C}$ be a sample space and $\mathcal{B}$ be a σ–field of subsets of $\mathcal{C}$. For fixed $B \in \mathcal{B}$ define the collection of subsets

$$\mathcal{B}_B = \{D \subset \mathcal{C} \mid D = E \cap B \text{ for some } E \in \mathcal{B}\}.$$

Show that $\mathcal{B}_B$ is a σ–field.

Note: In this case, the complement of $D \in \mathcal{B}_B$ has to be understood as $B \backslash D$; that is, the complement relative to B. The σ–field $\mathcal{B}_B$ is the σ–field $\mathcal{B}$ restricted to B, or *conditioned on* B.

***Solution**:* We verify that $\mathcal{B}_B$ satisfies the three properties of a σ–field.

(i) Observe that $\emptyset = \emptyset \cap B$, where $\emptyset \in \mathcal{B}$. Thus, $\emptyset \in \mathcal{B}_B$.

(ii) Let $D \in \mathcal{B}_B$. Then,

$$D = E \cap B \qquad \text{for some } E \in \mathcal{B}.$$

Then, the complement of D relative to B is

$$\begin{aligned} B \backslash D &= B \backslash (E \cap B) \\ &= B \cap (E \cap B)^c \\ &= B \cap (E^c \cup B^c) \\ &= (B \cap E^c) \cup (B \cap B^c) \\ &= (B \cap E^c) \cup \emptyset \\ &= B \cap E^c. \end{aligned}$$

Thus, $B \backslash D = E^c \cap B$, where $E^c \in \mathcal{B}$. It then follows that $B \backslash D \in \mathcal{B}_B$.

(iii) Let $D_1, D_2, D_3, \ldots$ denote a sequence of events in $\mathcal{B}_B$. Then, there exists a sequence $E_1, E_2, E_3, \ldots$ in $\mathcal{B}$ such that $D_k = E_k \cap B$ for all $k = 1, 2, 3, \ldots$
Then, by the distributive law,

$$\begin{aligned} \bigcup_{k=1}^{\infty} &= \bigcup_{k=1}^{\infty}(E_k \cap B) \\ &= \left(\bigcup_{k=1}^{\infty} E_k\right) \cap B, \end{aligned}$$

where

$$\bigcup_{k=1}^{\infty} E_k \in \mathcal{B}.$$

It then follows that

$$\bigcup_{k=1}^{\infty} D_k \in \mathcal{B}_B.$$

□

4. Let $\mathcal{S}$ denote the collection of all bounded, open intervals (a, b), where a and b are real numbers with $a < b$. Show that

$$\mathcal{B}(\mathcal{S}) = \mathcal{B}_o;$$

that is, the σ–field generated by bounded open intervals is the Borel σ–field.

Hints:

- We have already seen in the lecture that $\mathcal{B}_o$ contains all bounded open intervals.
- Observe also that the semi–infinite open interval (b, ∞) can be expressed as the union of the sequence of bounded intervals (b, k), for $k = 1, 2, 3, \ldots$

Proof: Let $\mathcal{S}$ denote the collection of all bounded, open intervals (a, b), for $a, b \in \mathbb{R}$ with $a < b$. We have proved in the lectures that

$$\mathcal{S} \subseteq \mathcal{B}_o.$$

Since $\mathcal{B}(\mathcal{S})$ is the smallest σ–algebra which contains $\mathcal{S}$, it follows that

$$\mathcal{B}(\mathcal{S}) \subseteq \mathcal{B}_o.$$

To show the reverse inclusion, first observe that, for any $b \in \mathbb{R}$,

$$(b, \infty) = \bigcup_{k=1}^{\infty} (b, k),$$

so that $(b, \infty) \in \mathcal{B}(\mathcal{S})$ for all $b \in \mathbb{R}$. It then follows that

$$(-\infty, b] = (b, \infty)^c \in \mathcal{B}(\mathcal{S}) \quad \text{for all } b \in \mathbb{R}.$$

Since intervals of the form $(-\infty, b]$ generate the Borel σ–field $\mathcal{B}_o$, it follows that

$$\mathcal{B}_o \subseteq \mathcal{B}(\mathcal{S}).$$

Combining this inclusion with the reverse inclusion that has been previous shown, we get that

$$\mathcal{B}(\mathcal{S}) = \mathcal{B}_o.$$

□

5. Show that for every real number a, the singleton $\{a\}$ is in the Borel σ–field $\mathcal{B}_o$.

   Hint: Express $\{a\}$ as an intersection of a sequence of open intervals.

   Proof: We have seen in the previous problem that $\mathcal{B}_o$ is also generated by the bounded, open intervals of the form (a, b). Thus, in view of Problem (1) in this set, we can prove that $\{a\}$ is in $\mathcal{B}_o$ by expressing it as an intersection of a sequence of such intervals.

   Consider the intervals

$$E_k = \left(a - \frac{1}{k}, a + \frac{1}{k}\right), \quad \text{for } k = 1, 2, 3, \ldots$$

   We claim that

$$\{a\} = \bigcap_{k=1}^{\infty} E_k.$$

   To see why this is so, let $x \in \bigcap_{k=1}^{\infty} E_k$. Then, $x \in E_k$ for all k; that is,

$$a - \frac{1}{k} < x < a + \frac{1}{k} \quad \text{for all } k.$$

   Since

$$\lim_{k\to\infty} \left(a - \frac{1}{k}\right) = \lim_{k\to\infty} \left(a + \frac{1}{k}\right) = a,$$

   it follows from the Sandwich Theorem that $x = a$. This proves the claim. □