Solutions to Assignment #3

1. *[Exercise 3 on page 18 in the text]*

   Consider two events A and B such that $\Pr(A) = 1/3$ and $\Pr(B) = 1/2$. Determine the value of $\Pr(B \cap A^c)$ for each of the following conditions:

   (a) A and B are disjoint;

   (b) $A \subseteq B$;

   (c) $\Pr(A \cap B) = 1/8$.

   Solution:

   If A and B are disjoint, then $B \cap A^c = B$. Thus,

   $$\Pr(B \cap A^c) = \Pr(B) = 1/2. \qquad \square$$

   If $A \subseteq B$, then

   $$\Pr(B \cap A^c) = \Pr(B \setminus A) = \Pr(B) - \Pr(A) = 1/2 - 1/3 = 1/6. \qquad \square$$

   From $B = (B \cap A) \cup (B \cap A^c)$, where the sets $B \cap A$ and $B \cap A^c$ are disjoint, we get that

   $$\Pr(B) = \Pr(B \cap A) + \Pr(B \cap A^c).$$

   Consequently,

   $$\Pr(B \cap A^c) = \Pr(B) - \Pr(A \cap B) = 1/2 - 1/8 = 3/8.$$

   $\square$

2. *[Exercise 7 on page 18 in the text]*

   Consider two events A and B with $\Pr(A) = 0.4$ and $\Pr(B) = 0.7$. Determine the maximum and minimum possible values for $\Pr(A \cap B)$ and the conditions under which each of these values is attained.

   Solution: If $A \subseteq B$ then, $A \cap B = A$ and

   $$\Pr(A \cap B) = \Pr(A) = 0.4.$$

On the other hand, from

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B),$$

and the fact that the largest value that $\Pr(A \cup B)$ can have is 1 we obtain that the smallest value that $\Pr(A \cap B)$ can have is

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) = 0.4 + 0.7 - 1 = 0.1.$$

Thus,

$$0.1 \leqslant \Pr(A \cap B) \leqslant 0.4.$$

The largest value occurs when $A \subseteq B$, and the smallest value occurs when $\Pr(A \cup B) = 1$. □

3. *[Exercise 9 on page 18 in the text]*

   Prove that for every two events A and B, the probability that exactly one of the two events will occur is given by the expression

   $$\Pr(A) + \Pr(B) - 2\Pr(A \cap B.$$

   Proof. Here we are talking about the event $(A \cup B) \setminus (A \cap B)$; that is, either A or B will occur, but not both simultaneously.

   Since $A \cap B \subseteq A \cup B$, we have that

   $$\begin{aligned} \Pr((A \cup B) \setminus (A \cap B)) &= \Pr(A \cup B) - \Pr(A \cap B) \\ &= \Pr(A) + \Pr(B) - \Pr(A \cap B) - \Pr(A \cap B) \\ &= \Pr(A) + \Pr(B) - 2\Pr(A \cap B). \end{aligned}$$

   □

4. Let A and B be elements in a σ–field $\mathcal{B}$ on a sample space $\mathcal{C}$, and let Pr denote a probability function defined on $\mathcal{B}$. Recall that $A \setminus B = \{x \in A \mid x \notin B\}$. Prove that if $B \subseteq A$, then

   $$\Pr(A \setminus B) = \Pr(A) - \Pr(B).$$

Proof. Write $A = B \cap (A \setminus B)$, where B and $A \setminus B$ are disjoint. Then,

$$\Pr(A) = \Pr(B) + \Pr(A \setminus B),$$

from which we get that

$$\Pr(A \setminus B) = \Pr(A) - \Pr(B).$$

$\square$

5. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and B an event in $\mathcal{B}$ with $\Pr(B) > 0$. Let
$$\mathcal{B}_B = \{D \subset \mathcal{C} \mid D = E \cap B \text{ for some } E \in \mathcal{B}\}.$$
We have already seen that $\mathcal{B}_B$ is a σ–field.

Let $P_{_B} \colon \mathcal{B}_B \to \mathbb{R}$ be defined by $P_{_B}(A) = \dfrac{\Pr(A)}{P(B)}$ for all $A \in \mathcal{B}_B$. Verify that $(B, \mathcal{B}_B, P_{_b})$ is a probability space; that is, show that $P_{_B} \colon \mathcal{B}_B \to \mathbb{R}$ is a probability function.

***Solution**:* First, observe that if $A \in \mathcal{B}_B$, then $A = E \cap B$ for some $E \in \mathcal{B}$, then, since $E \cap B \subseteq B$,

$$\Pr(E \cap B) \leqslant \Pr(B).$$

It then follows that $\Pr(A) \leqslant \Pr(B)$, from which we get that

$$\frac{\Pr(A)}{\Pr(B)} \leqslant 1.$$

It then follows that
$$P_{_B} \colon \mathcal{B}_B \to [0, 1].$$

Next, note that $P_{_B}(B) = \dfrac{\Pr(B)}{\Pr(B)} = 1.$

Finally, if $D_1, D_2, D_3, \ldots$ is a sequence of mutually disjoint sets in $\mathcal{B}_B$, then
$$\Pr\left(\bigcup_{k=1}^{\infty} D_k\right) = \sum_{k=1}^{\infty} \Pr(D_k),$$

since Pr is a probability function. It then follows that

$$
\begin{aligned}
P_{_B}\left(\bigcup_{k=1}^{\infty} D_k\right) &= \frac{1}{\Pr(B)}\Pr\left(\bigcup_{k=1}^{\infty} D_k\right) \\
&= \frac{1}{\Pr(B)}\sum_{k=1}^{\infty}\Pr(D_k) \\
&= \sum_{k=1}^{\infty}\frac{\Pr(D_k)}{\Pr(B)} \\
&= \sum_{k=1}^{\infty}P_{_B}(D_k).
\end{aligned}
$$

It then follows that $P_{_B}$ is indeed a probability function on $\mathcal{B}_B$. $\square$