Solutions to Assignment #4

1. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ be a sample space. Suppose that $E_1, E_2, E_3, \ldots$ is a sequence of events in $\mathcal{B}$ satisfying
$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots .$$
Then, $\lim_{n\to\infty} \Pr\,(E_n) = \Pr\left(\bigcap_{k=1}^{\infty} E_k\right)$.

   Hint: Use the analogous result for an increasing nested sequence of events presented in class and De Morgan's laws.

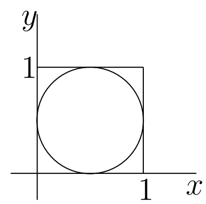
   Proof: Consider the sequence
$$E_1^c \subseteq E_2^c \subseteq E_3^c \subseteq \cdots .$$
By what was proved in class,
$$\lim_{n\to\infty} \Pr(E_n^c) = \Pr\left(\bigcup_{k=1}^{\infty} E_k^c\right),$$
where
$$\Pr(E_n^c) = 1 - \Pr(E_n), \quad \text{for all } \ n = 1, 2, 3, \ldots$$
so that
$$\lim_{n\to\infty} \Pr(E_n^c) = 1 - \lim_{n\to\infty} \Pr(E_n).$$
Now, by De Morgan's laws,
$$\begin{aligned} \Pr\left(\bigcup_{k=1}^{\infty} E_k^c\right) &= \Pr\left(\left(\bigcap_{k=1}^{\infty} E_k\right)^c\right) \\ &= 1 - \Pr\left(\bigcap_{k=1}^{\infty} E_k\right). \end{aligned}$$
It then follows that $\lim_{n\to\infty} \Pr\,(E_n) = \Pr\left(\bigcap_{k=1}^{\infty} E_k\right)$. □

2. *[Exercise 11 on page 18 in the text]*

   A point (x, y) is to be selected at random form a square S containing all the points (x, y) such that $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. Suppose that the probability that the selected point will belong to each specified subset of S is equal to the area of that subset. Find the probability of each of the following subsets:

(a) the subset of points such that $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \geqslant \frac{1}{4}$;

Solution: The event, E, in question here consists of all points inside the square S and outside the circle of radius 1/2 and center $(1/2, /1/2)$.



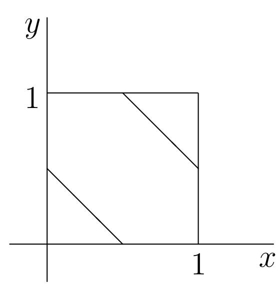
It then follows that

$$\Pr(E) = 1 - \pi\left(\frac{1}{2}\right)^2 = 1 - \frac{\pi}{4}.$$

□

(b) the subset of points such that $\frac{1}{2} < x + y < \frac{3}{2}$;

Solution: In this case, E is the portion of the square between the lines $x + y = 1/2$ and $x + y = 3/2$.

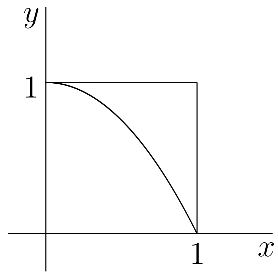


We obtain the area of E by subtracting from the area of S the areas of the two corner triangles pictured above; that is,

$$\Pr(E) = 1 - 2\left(\frac{1}{2}(0.5)(0.5)\right) = \frac{3}{4}.$$

□

(c) the subset of points such that $y < 1 - x^2$;



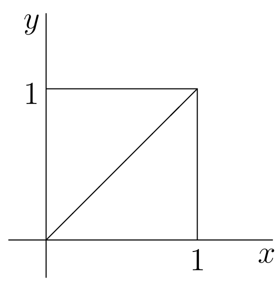
Solution: Here, E is the region in the square S under the graph of the parabola $y = 1 - x^2$.
It then follows that

$$\Pr(E) = \int_0^1 (1 - x^2)\ \ \mathrm{d}x = \frac{2}{3}.$$

□

(d) the subset of points such that $x = y$.

Solution: In this problem, E is the diagonal of the square S pictured below, which has zero area.



Hence, $\Pr(E) = 0$. □

3. *[Exercises 2 and 3 on page 22 in the text]*

- *Exercise 2.* If two balanced dice are rolled, what is the probability that the sum of the two numbers that appear will be even?

  Solution: The sample space for this experiment is listed in the table below

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$
$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$
$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$
$(6,1)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$

where the first entry in the pair (m, n) indicates number on the first die, and the second entry that on the second die. Since the dice are balanced, the outcomes are equally likely. In the next table the outcomes that yield an even sum are underlined

$\underline{(1,1)}$	$(1,2)$	$\underline{(1,3)}$	$(1,4)$	$\underline{(1,5)}$	$(1,6)$
$(2,1)$	$\underline{(2,2)}$	$(2,3)$	$\underline{(2,4)}$	$(2,5)$	$\underline{(2,6)}$
$\underline{(3,1)}$	$(3,2)$	$\underline{(3,3)}$	$(3,4)$	$\underline{(3,5)}$	$(3,6)$
$(4,1)$	$\underline{(4,2)}$	$(4,3)$	$\underline{(4,4)}$	$(4,5)$	$\underline{(4,6)}$
$\underline{(5,1)}$	$(5,2)$	$\underline{(5,3)}$	$(5,4)$	$\underline{(5,5)}$	$(5,6)$
$(6,1)$	$\underline{(6,2)}$	$(6,3)$	$\underline{(6,4)}$	$(6,5)$	$\underline{(6,6)}$

They represent half of all the outcomes; therefore, the probability that the sum of the numbers is even is $1/2$. □

- *Exercise 3.* If two balanced dice are rolled, what is the probability that the difference of the two numbers that appear will be less than 3?

  Solution: In the table below, that outcomes that yield a difference less than 3 are underlined

$\underline{(1,1)}$	$\underline{(1,2)}$	$\underline{(1,3)}$	$(1,4)$	$(1,5)$	$(1,6)$
$\underline{(2,1)}$	$\underline{(2,2)}$	$\underline{(2,3)}$	$\underline{(2,4)}$	$(2,5)$	$(2,6)$
$\underline{(3,1)}$	$\underline{(3,2)}$	$\underline{(3,3)}$	$\underline{(3,4)}$	$\underline{(3,5)}$	$(3,6)$
$(4,1)$	$\underline{(4,2)}$	$\underline{(4,3)}$	$\underline{(4,4)}$	$\underline{(4,5)}$	$\underline{(4,6)}$
$(5,1)$	$(5,2)$	$\underline{(5,3)}$	$\underline{(5,4)}$	$\underline{(5,5)}$	$\underline{(5,6)}$
$(6,1)$	$(6,2)$	$(6,3)$	$\underline{(6,4)}$	$\underline{(6,5)}$	$\underline{(6,6)}$

There are 24 of those outcomes. It then follows that the probability that the difference of the two numbers that appear will be less than 3 is

$$\frac{24}{36} \quad \text{or} \quad \frac{2}{3}.$$

□

4. A coin is tossed as many times as necessary to turn up one head. Thus, the elements of the sample space $\mathcal{C}$ corresponding to this experiment are

$$H, TH, TTH, TTTH, \ldots$$

Let Pr be a functions that assigns to these elements the values $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ respectively.

(a) Show that $\Pr(\mathcal{C}) = 1$.

Solution:

$$\Pr(\mathcal{C} = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1/2}{1 - 1/2} = 1.$$

□

(b) Let E_1 denote the event $E_1 = \{H, TH, TTH, TTTH \text{ or } TTTTH\}$, and compute $\Pr(E_1)$.

Solution:

$$\Pr(E_1) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}.$$

□

(c) Let $E_2 = \{TTTTH, TTTTTH\}$, and compute $\Pr(E_2)$, $\Pr(E_1 \cap E_2)$ and $\Pr(E_2 \setminus E_1)$

Solution: First, compute the sets

$$E_1 \cap E_2 = \{TTTTH\},$$

and

$$E_2 \setminus E_1 = \{TTTTTH\}.$$

Then,

$$\Pr(E_2) = \frac{1}{32} + \frac{1}{64} = \frac{3}{64},$$

$$\Pr(E_1 \cap E_2) = \frac{1}{32},$$

and

$$\Pr(E_2) = \frac{1}{64}.$$

□

5. Let $\mathcal{C} = \{x \in \mathbb{R} \mid x > 0\}$ and define Pr on open intervals (a, b) with $0 < a < b$ by

$$\Pr((a,b)) = \int_a^b e^{-x} \ \ \mathrm{d}x.$$

(a) Show that $\Pr(\mathcal{C}) = 1$.

Solution:

$$\Pr(\mathcal{C}) = \int_0^\infty e^{-x} \ \ \mathrm{d}x = -e^{-x}\Big|_0^\infty = 1.$$

□

(b) Let $E = \{x \in \mathcal{C} \mid 4 < x < \infty\}$, and compute $\Pr(E)$, $\Pr(E^c)$ and $\Pr(E \cup E^c)$.

Solution: $E^c = \{x \in \mathcal{C} \mid 0 < x \leqslant 4$ and

$$\Pr(E) = \int_4^\infty e^{-x} \ \ \mathrm{d}x = -e^{-x}\Big|_4^\infty = e^{-4},$$

$$\Pr(E^c) = \int_0^4 e^{-x} \ \ \mathrm{d}x = -e^{-x}\Big|_0^4 = 1 - e^{-4},$$

and

$$\Pr(E \cup E^c) = \Pr(\mathcal{C}) = 1.$$

□