

## Solutions to Assignment #5

1. Let  $(\mathcal{C}, \mathcal{B}, \Pr)$  be a probability space. Prove that if  $E_1$  and  $E_2$  are independent events in  $\mathcal{B}$ , then so are  $E_1$  and  $E_2^c$ .

*Hint:* Observe that  $E_1 \setminus E_2$  is a subset of  $E_1$ .

*Proof:* Assume that  $E_1$  and  $E_2$  are independent; then,

$$\Pr(E_1 \cap E_2) = \Pr(E_1) \cdot \Pr(E_2).$$

Next, write  $E_1 = (E_1 \setminus E_2) \cup (E_1 \cap E_2)$ , where

$$E_1 \setminus E_2 = E_1 \cap E_2^c$$

and

$$\begin{aligned} E_1 \setminus (E_1 \setminus E_2) &= E_1 \cap (E_1 \cap E_2^c)^c \\ &= E_1 \cap (E_1^c \cup E_2) \\ &= (E_1 \cap E_1^c) \cup (E_1 \cap E_2) \\ &= \emptyset \cup (E_1 \cap E_2) \\ &= E_1 \cap E_2 \end{aligned}$$

are mutually disjoint. It then follows that

$$\Pr(E_1) = \Pr(E_1 \cap E_2^c) + \Pr(E_1 \cap E_2),$$

or

$$\Pr(E_1 \cap E_2^c) = \Pr(E_1) - \Pr(E_1 \cap E_2).$$

Thus, since  $E_1$  and  $E_2$  are assumed to be independent,

$$\begin{aligned} \Pr(E_1 \cap E_2^c) &= \Pr(E_1) - \Pr(E_1) \cdot \Pr(E_2) \\ &= \Pr(E_1) \cdot [1 - \Pr(E_2)] \\ &= \Pr(E_1) \cdot \Pr(E_2^c), \end{aligned}$$

which shows that  $E_1$  and  $E_2$  are independent. □

2. [Exercises 1 and 2 on page 55 in the text]

- [Exercises 1] If  $A \subseteq B$  with  $\Pr(B) > 0$ , what is the value of  $\Pr(A \mid B)$ ?

**Solution:** If  $A \subseteq B$ , then  $A \cap B = A$ ; so that,

$$\Pr(A | B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)}.$$

□

- [Exercises 2] If  $A$  and  $B$  are disjoint events and  $\Pr(B) > 0$ , what is the value of  $\Pr(A | B)$ ?

**Solution:** If  $A \cap B = \emptyset$ , then

$$\Pr(A | B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{0}{\Pr(B)} = 0.$$

□

3. [Exercise 5 on page 55 in the text]

A box contains  $r$  red balls and  $b$  blue balls. One ball is selected at random and the color is observed. The ball is then returned to the the box and  $k$  additional balls of the same color are also put in the box. A second ball is the selected at random, its color is observed, and it is returned to the box with  $k$  additional balls of the same color. Each time another ball is selected, the process is repeated. If four balls are selected, what is the probability that the first three balls will be red and the fourth one will be blue?

**Solution:** Let  $R_1$  denote the event that the first ball that is selected is red. Then,

$$\Pr(R_1) = \frac{r}{r + b}.$$

Next, let  $R_2$  denote the event that the second ball that is selected is red. Then,

$$\Pr(R_1 \cap R_2) = \Pr(R_1) \cdot \Pr(R_2 | R_1),$$

where

$$\Pr(R_2 | R_1) = \frac{r + k}{r + k + b},$$

since  $k$  additional red balls have been added after the first draw. Then

$$\Pr(R_1 \cap R_2) = \frac{r}{r + b} \cdot \frac{r + k}{r + k + b}.$$

Similarly, if  $R_3$  denotes the event that the third selected ball is red,

$$\Pr(R_1 \cap R_2 \cap R_3) = \Pr(R_1 \cap R_2) \cdot \Pr(R_3 \mid R_1 \cap R_2),$$

where

$$\Pr(R_3 \mid R_1 \cap R_2) = \frac{r + 2k}{r + 2k + b}.$$

Thus,

$$\Pr(R_1 \cap R_2 \cap R_3) = \frac{r}{r + b} \cdot \frac{r + k}{r + k + b} \cdot \frac{r + 2k}{r + 2k + b}.$$

Finally, if  $B_4$  denotes the event that the fourth ball is blue, then

$$\Pr(R_1 \cap R_2 \cap R_3 \cap B_4) = \Pr(R_1 \cap R_2 \cap R_3) \cdot \Pr(B_4 \mid R_1 \cap R_2 \cap R_3),$$

where

$$\Pr(B_4 \mid R_1 \cap R_2 \cap R_3) = \frac{b}{r + 3k + b}.$$

It then follows that

$$\Pr(R_1 \cap R_2 \cap R_3 \cap B_4) = \frac{r}{r + b} \cdot \frac{r + k}{r + k + b} \cdot \frac{r + 2k}{r + 2k + b} \cdot \frac{b}{r + 3k + b}.$$

□

4. [Exercise 11 on page 55 in the text]

For any three events  $A$ ,  $B$  and  $D$ , such that  $\Pr(D) > 0$ , prove that

$$\Pr(A \cup B \mid D) = \Pr(A \mid D) + \Pr(B \mid D) - \Pr(A \cap B \mid D).$$

**Solution:** We have seen that the function

$$P_D(E \cap D) = \frac{\Pr(E \cap D)}{\Pr(D)} = \Pr(E \mid D),$$

defines a probability function on the  $\sigma$ -field  $\mathcal{B}_D$  made up of intersections of events with  $D$ . It then follows that

$$P_D((A \cup B) \cap D) = P_D(A \cap D) + P_D((B \cap D) \cap D) - P_D((A \cap B) \cap D),$$

from which the result follows.

□

5. [*The Monte Hall Problem*]. In a game show, suppose there are three curtains. Behind one curtain is a nice prize while behind the other two there are worthless prizes. A contestant selects one curtain at random, and then Monte Hall (the game show host) opens one of the other two curtains to reveal a worthless prize. Hall then expresses the willingness to trade the curtain that the contestant has selected for the other curtain that has not been opened. Should the contestant switch curtains or stick with the one that she has? If she sticks with the one she has then the probability of winning the prize is  $1/3$ . Hence, to answer this question, you must determine the probability that she wins the prize given that she switches.

**Solution:** Let  $W$  denote the event that the contestant wins the prize and  $S$  the event that she switches. We want to compute  $\Pr(W \mid S)$ .

Observe that, if the contestant switches curtains, then she will win the prize only if she picked one of the two worthless prizes to begin with. This occurs with a probability of 2 out of 3. Thus,

$$\Pr(W \mid S) = \frac{2}{3}.$$

On the other hand,

$$\Pr(W \mid S^c) = \frac{1}{3}.$$

Thus, in order to increase the chances of winning the prize, the contestant must switch.  $\square$